第三章、布朗运动

- 回顾：马氏链/跳过程，S 可数，$\{X_t\}$，

$$P(X_{t+s}=j|X_t=i, X_{t_1}=i_1,\cdots,X_{t_n}=i_n)=p_{ij}(s).$$

- 离散型→ 连续型：$\mathbb{R}$；　(条件)分布列→ (条件)密度.
- 定义3.0.1. 若$\forall n\geqslant 1$, $0\leqslant t_1<\cdots<t_{n-1}<t$ 与$s>0$,

$$\underset{\sim\sim\sim}{p_{X_{t+s}|(X_{t_1},\cdots,X_{t_{n-1}},X_t)}\big(y\big|(x_1,\cdots,x_{n-1},x)\big)}$$

只依赖于s, x 与y, 则称$\{X_t\}$ 是(时齐的)马氏过程.
- 转移密度：$\star\star \triangleq p_s(x,y)$.
- 注：时齐vs 非时齐$p_{t,s}(x,y)$.
- 注：允许$X_0\equiv x$. 记为P_x.

  若初分布μ 有密度$\rho(x)$, 则$P_\mu=\int P_x(A)\rho(x)\mathrm{d}x$.
- 例，P_x 的有限维联合密度：$p_{t_1,\cdots,t_n}(x_1,\cdots,x_n)$

$$=p_{t_1}(x,x_1)p_{t_2-t_1}(x_1,x_2)\cdots p_{t_n-t_{n-1}}(x_{n-1},x_n).$$

§3.1 高斯分布与高斯过程

- n 维正态分布$N(\vec{m}, \mathbf{\Sigma})$, 其中, $\mathbf{\Sigma}$ 正定. 联合密度:

$$\frac{1}{\sqrt{(2\pi)^n|\mathbf{\Sigma}|}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{m})\mathbf{\Sigma}^{-1}(\vec{x}-\vec{m})^T\right\}.$$

- n 维高斯分布$N(\vec{m}, \mathbf{\Sigma})$, 其中, $\mathbf{\Sigma}$ 半正定. 特征函数:

$$\exp\left\{\sqrt{-1}\vec{m}\cdot\vec{t}-\frac{1}{2}\vec{t}\,\mathbf{\Sigma}\,\vec{t}^T\right\}.$$

- 概率论:

  线性变换、子向量、条件分布、独立性iff 不相关.

- 高斯系/高斯过程$\mathbf{X}=\{X_\alpha : \alpha \in I\}$:

$$(X_{\alpha_1}, \cdots, X_{\alpha_n}) \text{ 服从高斯分布}, \quad \forall n; \alpha_1, \cdots, \alpha_n \in I.$$

- 命题3.1.2(独立iff 不相关). 设$\{X_\alpha : \alpha \in I\}$ 是高斯系, $I_1, \cdots, I_n \subseteq I$, 互不相交. 记$\mathbf{X}_r = \{X_\alpha : \alpha \in I_r\}$. 若

$$\mathrm{Cov}(X_\alpha, X_\beta) = 0, \quad \forall r \neq s; \alpha \in I_r, \beta \in I_s,$$

则$\mathbf{X}_1, \cdots, \mathbf{X}_n$ 相互独立.

- 命题3.1.3(线性变换). 设$\{X_\alpha : \alpha \in I\}$ 是高斯系. 若

$$Y_\beta = c_1 X_{\alpha_1} + \cdots + c_n X_{\alpha_n}, \quad \forall \beta \in J,$$

则$\{Y_\beta : \beta \in J\}$ 是高斯系.

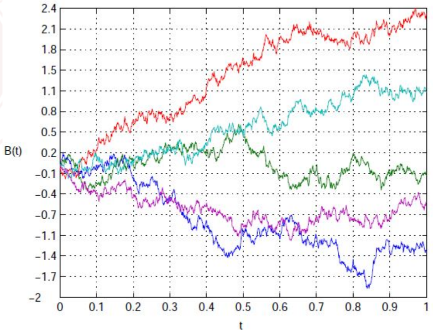
§3.2 布朗运动的定义与Levi 构造

一、定义

X_t = 时间t 之后的位移. 噪声.

(A1) X_t 各向同性.　　(A3) X_t 关于t 连续.

(A2) $X_{t+s} - X_t$ 与X_t 相互独立, 与X_s 具有相同的分布.



图片来源:“Brownian motion”, P. Mörters & Y. Peres.

定义

假设$B_0 = 0$. 若$\forall t \geqslant 0,\ s > 0,\ n \geqslant 2,\ \forall 0 < t_1 < \cdots < t_n$.

(B1) $B_{t+s} - B_t \sim N(0, s)$,

(B2) $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ 相互独立,

(B3) 轨道 a.s.连续:

$$\exists \Omega_0 \text{ s.t } P(\Omega_0) = 1 \text{且} \forall \omega \in \Omega_0, B_t(\omega) \text{关于} t \text{ 连续}.$$

则称$\{B_t : t \geqslant 0\}$ 为(一维标准)布朗运动(Brownian motion).

- 注: 也称$X_t = x + \sigma B_t$ 为BM.
- 注: 从x 出发: $\{x + B_t\} \sim P_x$;
  初分布为μ: $\{X + B_t\} \sim P_\mu$, 其中$X \sim \mu$, 与$\{B_t\}$ 独立.
- (B1) & (B2): 任意有限维边缘分布; 独立、平稳增量过程.
- (B3): 给定ω, 视为t 的函数. 轨道连续性.

- 命题3.2.2. $0 < t_1 < t_2 < \cdots < t_n$, $\left(B_{t_1}, B_{t_2}, \cdots, B_{t_n}\right)$ 服从n维正态分布, 其联合分布密度为

$$p_{t_1,t_2,\cdots,t_n}(x_1, x_2, \cdots, x_n) = \prod_{k=1}^{n} p_{t_k - t_{k-1}}(x_{k-1}, x_k),$$

其中$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(y-x)^2}{2t}\} = \frac{1}{\sqrt{t}}\phi\left(\frac{y-x}{\sqrt{t}}\right)$,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

- 证: 由(B1) & (B2) 可推出.
- 推论3.2.3. $\{B_t\}$ 是马氏过程.
- 注: 转移概率密度为$p_t(x,y)$.

命题 (命题3.2.4)

设

(C1) $\{X_t\}$ 是高斯过程,

(C2) $EX_t = 0$, $EX_tX_s = t \wedge s$.

(B3) $\{X_t\}$ 轨道连续.

则$\{X_t\}$ 是布朗运动.

- 证: $EB_t = 0$. $\forall t \leqslant s$,

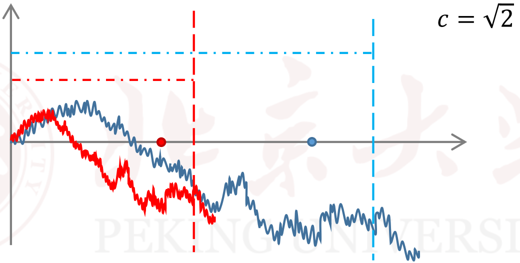
$$EB_tB_s = EB_t^2 + EB_t(B_s - B_t) = t.$$

- 注: (B1) & (B2) & (B3) $\Leftrightarrow$ (C1) & (C2) & (B3).

例3.2.5. 尺度变换性质. 时间= 空间^2. $(t = EB_t^2.)$

- $\forall c > 0$, $\{X_t\}$ 是标准布朗运动.

$$X_t = \frac{1}{c} B_{c^2 t}.$$



- (C1) $(X_{t_1}, \ldots, X_{t_n}) = \frac{1}{\sqrt{a}}(B_{at_1}, \ldots, B_{at_n})$, 为高斯向量.
- (C2) $EX_t = 0$,

$$EX_tX_s = \frac{1}{c^2} EB_{c^2t}B_{c^2s} = \frac{1}{c^2}(c^2t \wedge c^2s) = t \wedge s.$$

- (B3) 轨道连续.

例3.2.6. $\{W_t\}$ 是标准布朗运动.

$$W_t = \begin{cases} tB_{1/t}, & t > 0; \\ 0, & t = 0. \end{cases}$$

- (C1) 高斯过程, ✓.
- (C2) $EW_t = 0$. $EW_0W_s = 0$; $\forall 0 < t \leqslant s$,

$$EW_tW_s = EtB_{1/t} \times sB_{1/s} = ts \times EB_{1/t}B_{1/s} = ts \times \frac{1}{s} = t.$$

- (C3) 轨道连续, (将在推论3.4.9中完成).

  还需验证$\lim_{t\to 0+} W_t = \lim_{s\to\infty} B_s/s = 0$.
- $(0, 1, \infty) \leftrightarrow (\infty, 1, 0)$.

d 维标准布朗运动: $\vec{B}_t = (B_t^{(1)}, \ldots, B_t^{(d)})^T$.

- 验证$\{B_t^{(i)}\}$'s 是i.i.d.一维布朗运动:
  - (D1) $\{B_t^{(i)} : i, t\}$ 是高斯系;
  - (D2) $EB_t^{(i)} = 0$, $EB_t^{(i)} B_s^{(j)} = \mathbf{1}_{\{i=j\}} \cdot (t \wedge s)$,
  - (D3) 轨道连续.
- 命题3.2.8 (各向同性). 假设$\mathbf{O}$ 是d 维正交矩阵, 则$\{\mathbf{O}\vec{B}_t\}$ 仍然是d 维标准布朗运动.
  - 证: (D1) $\surd$. (D3) $\surd$. (D2): $EX_t^{(i)} = 0$, $\surd$.

$$EX_t^{(i)} X_s^{(j)} = E \underline{\sum_k o_{ik} B_t^{(k)}} \sum_\ell o_{j\ell} B_s^{(\ell)}$$

$$= \sum_{k,\ell} o_{ik} o_{j\ell} \mathbf{1}_{\{k=\ell\}} \cdot (t \wedge s) = \sum_k o_{ik} o_{jk} \cdot (t \wedge s) = \mathbf{1}_{\{i=j\}} \cdot (t \wedge s).$$

§3.2 习题5, 6.

- 转移概率密度: $p_t(\vec{x},\vec{y}) = \frac{1}{\sqrt{(2\pi t)^d}} e^{-\frac{\|\vec{y}-\vec{x}\|^2}{2t}}$.
- 格林函数: $G(\vec{x},\vec{y}) = \int_0^\infty p_t(\vec{x},\vec{y})\mathrm{d}t$. $G_{ij} = E_i V_j = \sum_{n=0}^\infty P_i(X_n = j)$
  - $G(\vec{x},\vec{y}) = C\int_0^\infty t^{-d/2} e^{-c/t}\mathrm{d}t = C\int_0^\infty s^{d/2-2} e^{-cs}\mathrm{d}s$, $\quad s = 1/t$.
  - $d = 1, 2 \Rightarrow s^{-1/2}, -\ln s$, $G(\vec{x},\vec{y}) = \infty$,

    $d = 3 \Rightarrow s^{d/2-1}$, $G(\vec{x},\vec{y}) < \infty$.
- C-K 等式: $p_{t+s}(\vec{x},\vec{y}) = \int_{\mathbb{R}^d} p_t(\vec{x},\vec{z})p_s(\vec{z},\vec{y})d\vec{z}$.
- Kolmogorov 前进、后退方程:

$$\frac{\partial p_t(\vec{x},\vec{y})}{\partial t} = \frac{1}{2}\Delta_y p_t(x,y) = \frac{1}{2}\Delta_x p_t(x,y),$$

$\Delta_x = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$.

二、Levi 构造

- 引理3.2.9. 设$X \sim N(0,\sigma^2)$. 若$\tilde{X}$ 与X i.i.d., 则

$$Y := \frac{1}{2}(X+\tilde{X}), \quad Z := \frac{1}{2}(X-\tilde{X})$$

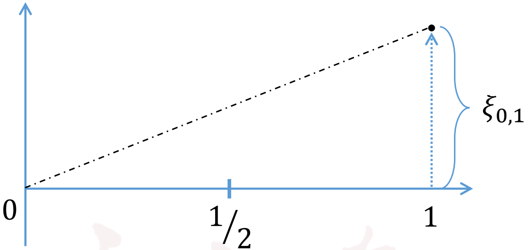
i.i.d., 都$\sim N(0,\sigma^2/2)$.

- 证:

$$Y := \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(X+\tilde{X}), \quad Z := \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(X-\tilde{X}),$$

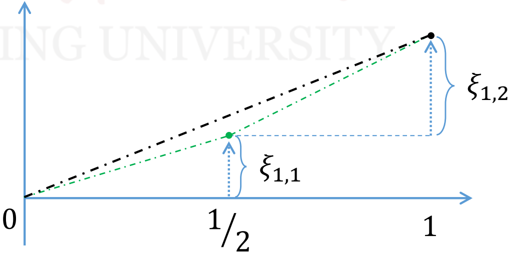
- 注: X 的分解, $X = Y + Z$.

- 任取$\xi = \xi_{0,1} \sim N(0,1)$.
  令$B_1 = \xi_{0,1}$.
  线性插值得到$B_t^{(0)}$.



- 取$\tilde{\xi} = \tilde{\xi}_{0,1}$. 令
  $\xi_{1,1} = \frac{1}{2}(\xi + \tilde{\xi})$,
  $\xi_{1,2} = \frac{1}{2}(\xi - \tilde{\xi})$.　于是$\xi = \xi_{1,1} \bigoplus \xi_{12}$. $B_{\frac{1}{2}} = \xi_{1,1}$.
- 令$B_{\frac{1}{2}} = \xi_{1,1} = \frac{1}{2}(\xi + \tilde{\xi})$.
  插值得到$B_t^{(1)}$.
- D_1:

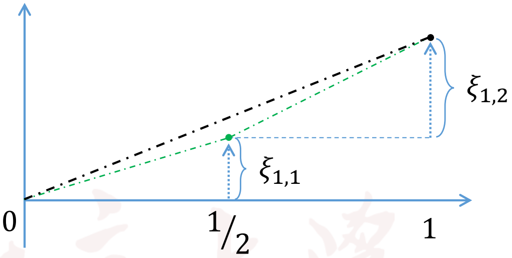
$$= \max_{0\leqslant t\leqslant 1} |B_t^{(1)} - B_t^{(0)}|$$
$$= |\tfrac{1}{2}\xi - \tfrac{1}{2}(\xi + \tilde{\xi})|$$
$$= \tfrac{1}{2}|\tilde{\xi}|.$$



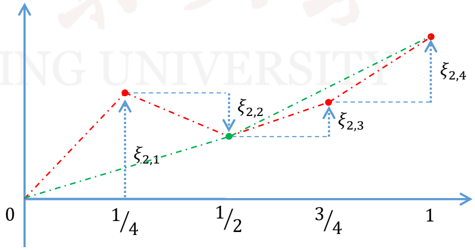
- 取$\tilde{\xi}_{1,1}, \tilde{\xi}_{1,2} \sim N(0, \frac{1}{2})$，使得

  $\xi_{11} = \xi_{2,1} \bigoplus \xi_{2,2}$,

  $\xi_{1,2} = \xi_{2,3} \bigoplus \xi_{2,4}$.



- 获得$B_{\frac{1}{4}}, B_{\frac{1}{2}}, B_{\frac{3}{4}}$.
  插值得到$B_t^{(2)}$.

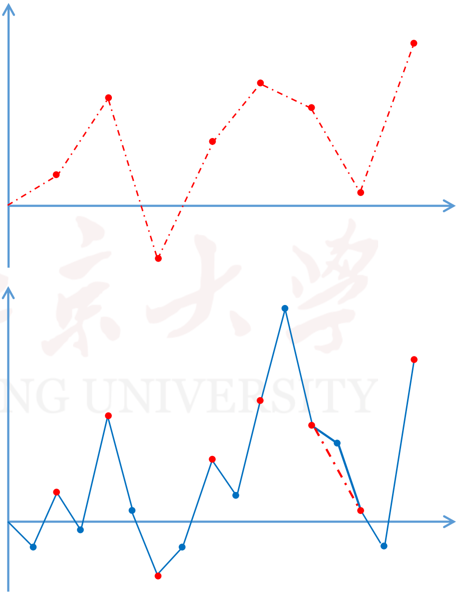


- D_2:

  $$= \max_{0\leqslant t\leqslant 1} |B_t^{(2)} - B_t^{(1)}|$$

  $$= \frac{1}{2} \max\{|\tilde{\xi}_{1,1}|, |\tilde{\xi}_{1,2}|\}.$$

- 假设已有$B_t^{(n)}$.
- 取$\tilde{\xi}_{n,i} \sim N(0, \frac{1}{2^n})$,
  获得$B_{\frac{i}{2^{n+1}}}$,
  插值得到$B_t^{(n+1)}$.
- D_n:
  $$= \max_{0\leqslant t\leqslant 1} |B_t^{(n+1)} - B_t^{(n)}|$$
  $$= \frac{1}{2} \max_{1\leqslant i\leqslant 2^n} |\tilde{\xi}_{n,i}|.$$
- 往证$D_n = O(2^{-n/4}\sqrt{n})$.



- $A_n := \left\{ \max_{1 \leqslant i \leqslant 2^n} |\tilde{\xi}_{n,i}| \geqslant 2^{-n/4}\sqrt{n} \right\}$, $n \geqslant 1$ 相互独立.
- $\sum_n P(A_n) < \infty$:

$$P(A_n) \leqslant 2^n P(\sqrt{2^{-n}}|Z| \geqslant 2^{-n/4}\sqrt{n})$$
$$= 2^n P(|Z| \geqslant 2^{n/4}\sqrt{n}) \leqslant 2^n \frac{EZ^4}{2^n n^2} = \frac{1}{n^2} EZ^4.$$

- Borel-Cantelli引理: $P(A_n\text{发生有限次}) = 1$.

  $\forall \omega$, $\exists N(\omega)$ s.t. $\forall n \geqslant N(\omega)$, $\omega \notin A_n$, 即

$$\max_{0 \leqslant t \leqslant 1} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)| \leqslant \frac{1}{2} \times 2^{-n/4}\sqrt{n}.$$

- $[0,1]$ 上的连续函数$\mathbf{B}^{(n)}(\omega) : t \mapsto B_t^{(n)}(\omega)$ 一致收敛到某连续函数$\mathbf{B}(\omega) : t \mapsto B_t(\omega)$. $\{B_t\}$ 为B.M.

三、总结

- B.M.的定义/验证: (B1)、(B2)、(B3).
- B.M.的验证: (C1)、(C2)、(B3).
- 独立、平稳增量过程.
- 马氏过程、转移概率.

§3.3* 不变原理(Invariant Principle)

- 随机变量之依分布收敛iff

$$\forall f : \mathbb{R} \to \mathbb{R}, \text{有界连续函数}, Ef(X_n) \to Ef(X).$$

- 固定T. $\{B_t : 0 \leqslant t \leqslant T\}$ 是随机轨道, 取值于$\mathbb{C}_T = C[0,T]$.
- $\mathbb{C}_T = C[0,T]$ 是一个完备可分的距离空间:

$$d_T(\varphi,\psi) := \max_{0\leqslant t\leqslant T} |\varphi(t) - \psi(t)|, \quad \forall \varphi,\psi \in C[0,T].$$

- 不变原理.

定理 (不变原理, 定理3.3.1)

$\forall T > 0$, $\forall f : C[0,T] \to \mathbb{R}$, 有界连续泛函,

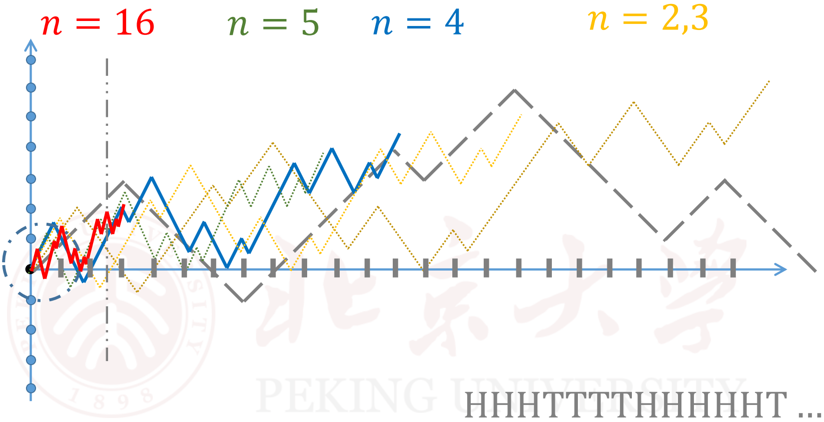
$$\lim_{n\to\infty} Ef\Big(\{\frac{1}{\sqrt{n}} S_{nt} : 0 \leqslant t \leqslant T\}\Big) = Ef\Big(\{B_t : 0 \leqslant t \leqslant T\}\Big).$$

不变原理的直观解释:

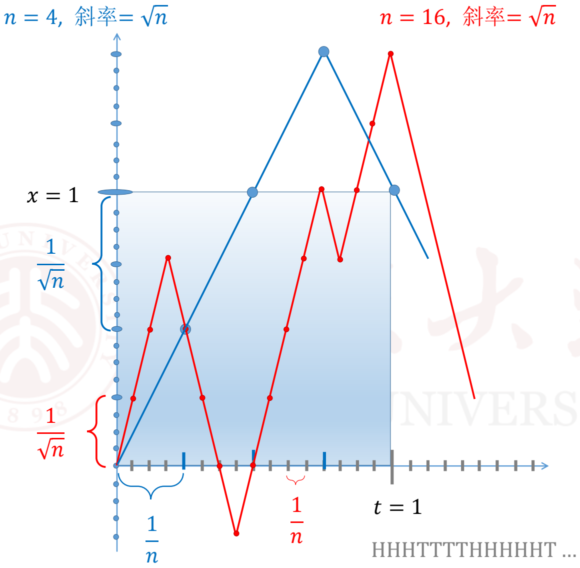
- CLT: $\xi_1, \xi_2, \cdots$ i.i.d., $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$, $S_m = \xi_1 + \cdots + \xi_m$. 则

$$\frac{1}{\sqrt{n}} S_n \xrightarrow{d} Z \sim N(0,1).$$

- 时间 $=$ 空间^2.
- $\Delta t = \frac{1}{n}$, $\Delta x = \frac{1}{\sqrt{n}}$. $\frac{1}{\sqrt{n}} S_n = \frac{1}{\sqrt{n}}\xi_1 + \ldots + \frac{1}{\sqrt{n}}\xi_n \overset{d}{\approx} B_1$.
- $t = m\Delta t$, 即$m = nt$, $\frac{1}{\sqrt{n}}\xi_1 + \ldots + \frac{1}{\sqrt{n}}\xi_m = \sqrt{t} \cdot \frac{1}{\sqrt{nt}} S_{nt} \xrightarrow{d} \sqrt{t} Z \overset{d}{=} B_t$.
- $\{\frac{1}{\sqrt{n}} S_{nt} : t \geqslant 0\}$ 独立平稳增量, 且增量的分布 $\overset{n\to\infty}{\longrightarrow} N(0,s)$.
- 插值: $\forall t \in [m, m+1]$, 令$S_t = S_m + (t-m)(S_{m+1} - S_m)$.



- $\frac{1}{\sqrt{n}}S_n \xrightarrow{d} Z \sim N(0,1)$, 但是a.s.-$\lim\limits_{n} \frac{1}{\sqrt{n}}S_n$ 不存在.
- §3.2 习题8. $\{B_{T-t}-B_T : 0 \leqslant t \leqslant T\}$, $\{-B_t : t \geqslant 0\}$ 为B.M., 强马氏性(定理3.4.1)、反射原理(推论3.4.2).



- 轨道处处不可微、$[0, \varepsilon)$ 中含无穷多个零点.

例3.3.4. $\frac{1}{\sqrt{n}}\max_{0\leqslant m\leqslant n} S_m \xrightarrow{d} \max_{0\leqslant t\leqslant 1} B_t$.

- 令$F: C([0,1]) \to \mathbb{R}$, $\varphi \mapsto \max_{0\leqslant t\leqslant 1}\varphi(t)$. 则$F$ 是连续函数.
- 若$d_T(\varphi,\psi) := \max_{0\leqslant t\leqslant T}|\varphi(t)-\psi(t)| \leqslant \varepsilon$, 则$|F(\varphi)-F(\psi)| \leqslant \varepsilon$:

$$F(\varphi) = \varphi(t_\varphi) \leqslant \psi(t_\varphi) + \varepsilon \leqslant \psi_{t_\psi} + \varepsilon = F(\psi) + \varepsilon.$$

- F 不是有界函数.
- 往证: $F\left(\{\frac{1}{\sqrt{n}}S_{nt} : 0 \leqslant t \leqslant 1\}\right) \xrightarrow{d} F\left(\{B_t : 0 \leqslant t \leqslant 1\}\right)$.
- $\forall f: \mathbb{R} \to \mathbb{R}$ 有界连续, $f\circ F: C([0,1]) \to \mathbb{R}$ 有界连续.

$$Ef\circ F\left(\{\frac{1}{\sqrt{n}}S_{nt} : 0 \leqslant t \leqslant 1\}\right) \to Ef\circ F\left(\{B_t : 0 \leqslant t \leqslant 1\}\right).$$

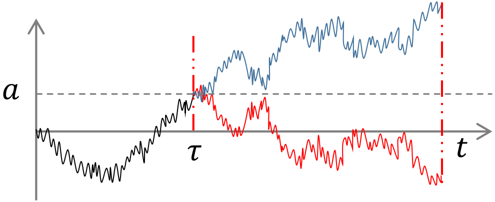
故, ✓.

一、首达时

- $\tau_a := \inf\{t \geqslant 0 : B_t = a\}$. $\tau_a^{(r)} := \inf\{t \geqslant r : B_t = a\}$.
- 定理3.4.1 (强马氏性). $\tau = \tau_a$ 或$\tau_a^{(r)}$. 在$\{\tau < \infty\}$ 的条件下, $\{B_{\tau+t} - B_\tau\}$ 为标准布朗运动, 且与τ 独立.
- 推论3.4.2 (反射原理).

$$P_0(\tau_a < t, B_t > a) = P_0(\tau_a < t, B_t < a).$$

- 证: 由强马氏性& 对称性可得.



命题 (命题3.4.3)

$\forall a > 0$, τ_a 是连续型. 分布函数与密度如下:

$$P_0(\tau_a \leqslant t) = 2P_0(B_t > a), \quad \forall t \geqslant 0;$$

$$p(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}, \quad t > 0.$$

- $P_0(\tau_a = t) \leqslant P_0(B_t = a) = 0$.
- 由反射原理, $P_0(\tau_a \leqslant t) = 2P_0(B_t > a)$.
- 密度: $P_0(\tau_a \leqslant t) = 2P_0(B_1 > \frac{a}{\sqrt{t}})$, 故

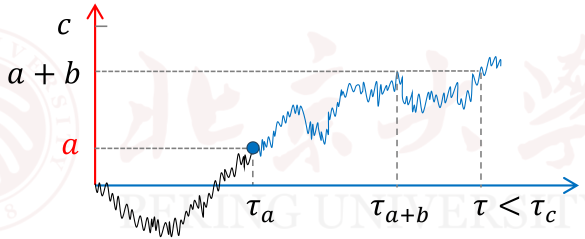
$$\rho_{\tau_a}(t) = -2\phi(\frac{a}{\sqrt{t}})\frac{\mathrm{d}\, a/\sqrt{t}}{\mathrm{d}t} = 2\phi(\frac{a}{\sqrt{t}})\frac{1}{2}\frac{a}{\sqrt{t^3}} = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}.$$

- 注: $P_0(\tau_a < \infty) = 1$, $E_0\tau_a = \infty$, $\forall a \neq 0$. (vs §3.4 习题14)

例3.4.5 & 3.4.6. 首达时过程: $\{\tau_a : a \geqslant 0\}$.

- 独立平稳增量, 但不满足(B3).

$$\tilde{B}_t = B_{\tau_a+t}. \qquad \tau_{a+b} - \tau_a = \tilde{\tau}_b \overset{d}{=} \tau_b, \text{ 且与}\tau_a\text{独立}.$$



- 尺度变换. $\{\hat{B}_t := B_{c^2t}/c\}$ 为B.M..

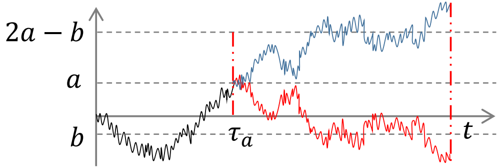
$$\hat{\tau}_a = \inf\{t : \hat{B}_t = a\} = \inf\{t : B_{c^2t} = ca\}$$

$$= \inf\{s/c^2 : B_s = ca\} = \tau_{ca}/c^2. \qquad \text{故, } \{\tau_{ca}/c^2\} \overset{d}{=} \{\tau_a\}.$$

二、最大值

- $M_t = \max_{0 \leqslant s \leqslant t} B_s$.
- 命题3.4.7. M_t 是连续型，且$M_t \stackrel{d}{=} |B_t|$.
- 证：$M_t > a$ iff $\tau_a < t$，概率为$P_0(|B_t| > a)$.
- 注：最大值在区间内部达到.
- §3.4. 习题4 & 5. (M_t, B_t) 是连续型，求：联合密度.
- 解：$\forall a > b \vee 0$,

$$P_0(M_t > a, B_t \leqslant b) = P_0(\tau_a \leqslant t, B_t > 2a - b) = P_0(B_t > 2a - b).$$



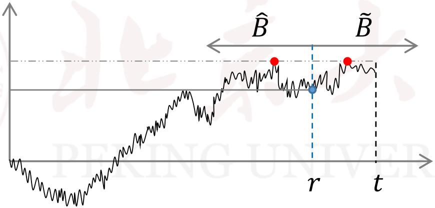
- §3.4. 习题7. $P_0(M_t > a | B_t = M_t)$: $(M_t, M_t - B_t)$.

推论 (推论3.4.8, 最大值点唯一)

$\forall t > 0,$

$$P_0(\underset{\sim}{\exists\ u < v < t,\ \text{s.t. } B_u = B_v = M_t}) = 0.$$

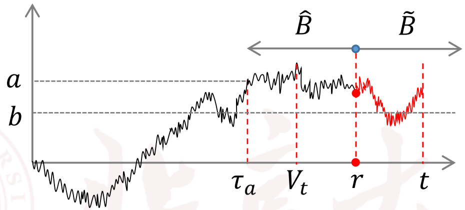
- $A_r = \{\exists\ u < r < v < t,\ \text{s.t. } B_u = B_v = M_t\}$.



- $\{\hat{B}_s = B_{r-s} - B_r\}$ 与$\{\tilde{B}_s = B_{s+r} - B_r\}$ 为独立的B.M..
- $P_0(A_r) = P(\hat{M}_r = \tilde{M}_{t-r}) = 0$
- $P(\underset{\sim}{\star\star}) \leqslant \sum_{r\in\mathbb{Q}_+} P_0(A_r) = 0.$

§3.4 习题6. V_t: 最大值点. 求(M_t, B_t, V_t) 的联合密度.

- $M_t > a, V_t \leqslant r$ iff $M_r > a,\ \tilde{M}_{t-r} < (\hat{M}_r =) M_r - B_r$.



- $\forall a > b \vee 0$, $M_t > a, V_t \leqslant r, B_t \leqslant b$ iff A, $\tilde{B}_{t-r} + B_r \leqslant b$.
- 记$p_r(x, y) = p_{(M_r, B_r)}(x, y)$, 则

$$P_0(\star\star) = P_0(AB)$$

$$= \iint p_r(x, y) \mathbf{1}_{\{x>a\}} P_0(M_{t-r} < x - y, B_{t-r} \leqslant b - y) \mathrm{d}x\mathrm{d}y.$$

推论 (推论3.4.9)

$P_0\left(\lim_{t\to\infty}\frac{B_t}{t}=0\right)=1.$

- SLLN: $\frac{B_n}{n}\to 0.$
- 令$D_n=\max_{n\leqslant s\leqslant n+1}|B_s-B_n|.$
- $D_0, D_1, D_2, \ldots$ i.i.d., $D_0\stackrel{d}{=}M_1\vee\hat{M}_1.$

$$ED_n\leqslant 2EM_1=2E|B_1|<\infty.$$

- $\frac{1}{n}D_n\xrightarrow{\text{a.s.}}0$ (推论0.3.2).
- 当$n\leqslant t<n+1$ 时, $|B_t|\leqslant|B_n|+D_n$. 故✓.

三、轨道性质

命题（命题3.4.10）

$P_0(\forall a < b, B|_{[a,b]}\text{ 不单调}) = 1.$

- 将$[a,b]$ 等分为n 段，分点为$t_0 = a < t_1 < \cdots < t_n = b$.
- $\{B|_{[a,b]}\text{ 单调}\} \triangleq A_{a,b}$. 则

$$A_{a,b} \subseteq \{B_{t_m} \geqslant B_{t_{m-1}}, \forall m \leqslant n\} \bigcup \{B_{t_m} \leqslant B_{t_{m-1}}, \forall m \leqslant n\}.$$

- $P(A_{a,b}) = 0$: $P(A_{a,b}) \leqslant 2 \times 2^{-n} \to 0$.
- $A = \sum_{r,s\in\mathcal{Q},0<r<s} A_{r,s}$. 故✓.

命题 (命题3.4.11)

布朗运动的轨道以概率1 处处不可微.

- 数学分析: 设φ 为$[0,1]$ 上(给定)的函数.
- $\varphi'(t_0)$ ∃: $\exists M>0$, $\exists \delta>0$, 使得

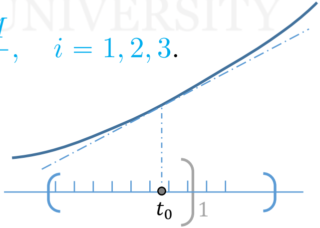
$$|\varphi(t)-\varphi(t_0)|\leqslant M|t-t_0|,\quad \forall\ t\in[t_0-\delta,t_0+\delta].$$

- $\forall n>\frac{4}{\delta}$, $\exists 0\leqslant k\leqslant n-3$ 使得

$$\left|\varphi(\frac{k+i}{n})-\varphi(\frac{k+i-1}{n})\right|\leqslant\frac{8M}{n},\quad i=1,2,3.$$

- $\{B_t$在 $[0,1]$ 中有可微点$\}\subseteq$

$$\bigcup_{M=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\bigcup_{k=0}^{n-3}A_{n,k}.$$



- $\{B_t$在 $[0,1]$ 中有可微点$\} \subseteq \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{k=0}^{n-3} A_{n,k}$.

$$A_{n,k} = \left\{ \left| B(\frac{k+i}{n}) - B(\frac{k+i-1}{n}) \right| \leqslant \frac{8M}{n},\ i = 1,2,3 \right\}.$$

- 往证$\lim_{n\to\infty} \sum_{k=0}^{n-3} P_0(A_{n,k}) = 0$, 于是✓.
- $Z \sim N(0,1)$, 密度为$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \leqslant \frac{1}{2}$.

$$P_0(A_{n,k}) = P(|\frac{1}{\sqrt{n}} Z| \leqslant \frac{8M}{n})^3 = P(|Z| \leqslant \frac{8M}{\sqrt{n}})^3.$$

$$P_0(|Z| \leqslant \frac{8M}{\sqrt{n}}) = \int_{-\frac{8M}{\sqrt{n}}}^{\frac{8M}{\sqrt{n}}} \phi(x) \mathrm{d}x \leqslant \frac{8M}{\sqrt{n}}.$$

- $\sum_{k=0}^{n-3} P_0(A_{n,k}) \leqslant n(8M/\sqrt{n})^3 = O(1/\sqrt{n}) \to 0.$

精细估计*.

- 设φ 为$[0,1]$ 上的函数. $\forall \delta > 0$, δ-最大振幅(oscillation):

$$\mathrm{osc}(\delta) := \max_{t,s\in[0,1],|t-s|\leqslant\delta} |\varphi(t)-\varphi(s)|.$$

- 注: 若φ 连续可微, 则$|\varphi(t)-\varphi(s)| = \varphi'(u)|t-s|$, 故 $\mathrm{osc}(\delta) \leqslant \max_{u\in[0,1]} |\varphi'(u)| \times \delta = O(\delta)$.
- 命题3.4.12. $\mathrm{osc}(\delta) = O\left(\sqrt{\delta \ln(\frac{1}{\delta})}\right)$, a.s..

$$P_0\left(\limsup_{\delta\to 0} \frac{\mathrm{osc}(\delta)}{\sqrt{\delta\ln(\frac{1}{\delta})}} \leqslant 6\right) = 1.$$

- 注3.4.13. $1 = (\frac{1}{\delta})^0 \ll \ln\frac{1}{\delta} \ll (\frac{1}{\delta})^{2\varepsilon}$.

  $\forall \alpha = \frac{1}{2} - \varepsilon \in (0, 1/2)$, 存在随机变量$\eta$ 使得

$$P_0\big(|B_t - B_s| \leqslant \eta\cdot|t-s|^\alpha,\ \forall t,s\in[0,1]\big) = 1.$$

  但此结论在$\alpha = \frac{1}{2} = \frac{1}{2} - 0$ 时不成立.

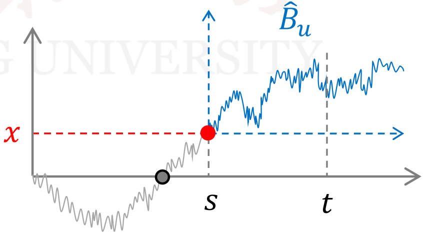
四、零点

命题 (反正弦律, 命题3.4.14)

令$L_t = \sup\{s \leqslant t : B_s = 0\}$. 其分布函数与密度如下:

$$P_0(L_t \leqslant s) = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}, \quad p(s) = \frac{1}{\pi\sqrt{t(t-s)}}, \quad \forall\ 0 \leqslant s \leqslant t.$$

- $P_0(L_t \leqslant s) = P\big(\hat{B}_u \neq -B_s, \forall u \in [0, t-s]\big)$.
- $P_0(\hat{\tau}_x > t - s)$
  $= P_0(|\hat{B}_{t-s}| \leqslant |x|)$.
- $P_0(L_t \leqslant s)$
  $= P(\sqrt{t-s}|W| \leqslant \sqrt{s}|Z|)$
  $= \frac{2}{\pi}\arcsin\frac{\sqrt{s}}{\sqrt{t}}$.



- 注: $P_0(L_t = 0) = 0$ vs $P_0(L_t > 0) = 1$.

推论 (推论3.4.15)

令$\sigma_0 := \inf\{t > 0 : B_t = 0\}$. 则$P_0(\sigma_0 = 0) = 1$.

- 证: $P_0(0 < L_{1/n} < 1/n) = 1$, 且$A := \bigcap_{n=1}^{\infty} A_n \subseteq \{\sigma_0 = 0\}$.
- 另证: $P_0(\sigma_0 \leqslant t) = P_0(L_t > 0) = 1$, $\forall t$. 故$P_0(\sigma_0 = 0) = 1$.
- 另证: $\exists$ t_n 介于$\tau_{1/n}$ 与$\tau_{-1/n}$ 之间, 使得$B_{t_n} = 0$.
  当$n \to \infty$ 时, $\tau_{\pm 1/n} \to \tau_0$. (§3.4 习题12)
- 注: 没有"首次返回原点"的时刻.

- 零点集:

$$\mathcal{Z} = \{t \geqslant 0 : B_t = 0\}, \quad \mathcal{Z}(\omega) = \{t \geqslant 0 : B_t(\omega) = 0\}.$$

- 去除概率为0 的轨道集Ω_0. 取定$\omega \notin \Omega_0$.
- $B_t(\omega)$ 关于t 连续. 故, $\mathcal{Z}(\omega)$ 是闭集:

$$t_1, t_2, \cdots \in \mathcal{Z}(\omega) \text{ 且 } \lim_n t_n = t \Rightarrow B_t(\omega) = \lim_n B_{t_n}(\omega) = 0 \Rightarrow t \in \mathcal{Z}(\omega).$$

- 设$D \subseteq \mathbb{R}$, 为闭集. 若$\forall t \in D$, $\exists t_1, t_2, \cdots \in D \backslash \{t\}$,
  使得 $\lim_n t_n = t$, 则称D 为完全集. (定义3.4.17)
- 例: $0 \in \mathcal{Z}(\omega)$, 且$L_{1/n}(\omega) > 0$ 且$\lim_n L_{1/n} = 0$.
- 注: $D' = \{t \geqslant 0 : \exists\ t_n \in D$ 使得 $\lim_n t_n = t\}$.
  D 是闭集, 故$D' \subseteq D$. 完全集: $D \subseteq D'$.

推论 (推论3.4.18)

$P_0(\mathcal{Z}$ 是完全集$) = 1.$

- 去除概率为0 的轨道集Ω_0. $\forall \omega \notin \Omega_0$, $B_t(\omega)$ 关于t 连续. 以下取定$\omega \in \Omega_0$.
- 轨道连续, 故$\mathcal{Z}(\omega)$ 是闭集, ✓.
- $\mathcal{Z}'(\omega) = \{t \geqslant 0 : \exists\ t_n \in \mathcal{Z}(\omega)$ 使得 $\lim_n t_n = t\}$.
- 往证: $\mathcal{Z}(\omega) \subseteq \mathcal{Z}'(\omega)$, 即$\forall t \in \mathcal{Z}(\omega)$, $\exists t_n \in \mathcal{Z}(\omega)$ 使得$\cdots\cdots$
- 关键: 如何描述$\forall t \in \mathcal{Z}(\omega)$?

- 取定ω. $\mathcal{Z}(\omega)=\{0\}\cup C_-(\omega)\cup C_+(\omega)$,

  $C_-(\omega):=\{t>0:\exists\ t_1,t_2,\cdots\in\mathcal{Z}(\omega)\cap[0,t)$ 使得 $\lim_n t_n=t\}$;

  $C_+(\omega):=\{t>0:B_t=0$ 且$\exists\delta>0$ 使得$B_s\neq 0,\forall s\in(t-\delta,t)\}$.

- $\{0\}$: 再去除概率为0 的集合$\tilde{\Omega}_0:=\{\sigma_0>0\}$.

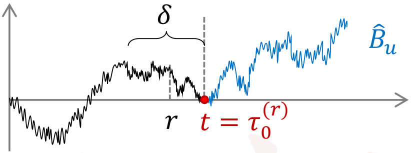
  $\forall\omega\notin\Omega_0\cup\tilde{\Omega}_0$, $\sigma_0(\omega)=0$, 故$0\in\mathcal{Z}'(\omega)$.

- $C_-(\omega)$: 按定义, $C_-(\omega)\subseteq\mathcal{Z}'(\omega)$, $\forall\omega\notin\Omega_0$.

- 以下, 处理$C_+(\omega)$.

- $\forall t \in C_+(\omega)$: $t > 0$, $B_t = 0$,

  $\exists \delta > 0$ 使得$B_s \neq 0, \forall s \in (t-\delta, t)\}$.



- $\exists r \in \mathbb{Q}_+ \cap (t-\delta, t)$, 使得

$$t = \inf\{s \geqslant r : B_s(\omega) = 0\} \triangleq \tau_0^{(r)}(\omega).$$

- $C_+(\omega) \subseteq \{\tau_0^{(r)}(\omega) : r \in \mathbb{Q}_+\}$.
- 固定$r \in \mathbb{Q}_+$. 记$\tau = \tau_0^{(r)}$, 则$\{\hat{B}_u = B_{\tau+u}(-B_\tau)\}$ 为布朗运动.
- 去除零概率集合Ω_r. $\forall \omega \notin \Omega_0 \cup \Omega_r$, $0 \in \mathcal{Z}'(\hat{\omega})$, 即$t \in \mathcal{Z}'(\omega)$.
- $\forall \omega \notin \Omega_0 \cup \bigcup_{r \in \mathbb{Q}_+} \Omega_r$,

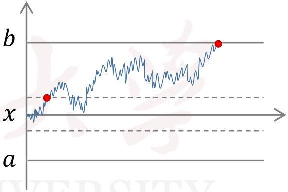
$$\tau_0^{(r)}(\omega) \in \mathcal{Z}'(\omega), \forall r \in \mathbb{Q}_+ \Rightarrow C_+(\omega) \subseteq \mathcal{Z}'(\omega).$$

五、总结

- 反射原理: $P_0(\tau_a < t, B_t > a) = P_0(\tau_a < t, B_t < a)$.
- 首达时、最大值: $\tau_a < t$ iff $M_t > a$, 分布、密度, $M_t \stackrel{d}{=} |B_t|$.
- 零常返、点常返.
- 最大值点V_t 唯一.
- $B_t/t \to 0$, $\{tB_{1/t}\}$ 是布朗运动.
- 振荡: 不单调、不可微.
- 最后一个零点L_t, $\sigma_0 = 0$.
- 零点集.

命题 (命题3.5.1)

$$P_x(\tau_b < \tau_a) = \frac{x-a}{b-a}, \quad \forall a \leqslant x \leqslant b.$$



- 令$\varphi(x) = P_x(\tau_b < \tau_a)$.
- 强马、对称性:
$$\varphi(x) = \frac{1}{2}(\varphi(x+\delta) + \varphi(x-\delta)).$$
- 边界条件: $\varphi(a) = 0$, $\varphi(b) = 1$. 故, $\forall[a, b]$ 的二分点, ✓.
- 单调性: $x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$. 故, ✓.
- 推论3.5.4(Wald 引理). $E_x B_{\tau_a \wedge \tau_b} = x$.
- 证: LHS $= b\varphi(x) + a(1-\varphi(x)) = x$.

- 例3.5.2. $\{S_n = B_{\sigma_n}\}$ 是SRW, 其中, $\sigma_0 = 0$,

$$\sigma_n := \inf\{t \geqslant \sigma_{n-1} : \left|B_t - B_{\sigma_{n-1}}\right| = 1\}.$$

- 不变原理: $N \gg 1$, $t = \frac{n}{N}$, ($\{S_n\}$ 线性插值),

$$\{S_t^{(N)} = S_{Nt}/\sqrt{N}\} \overset{d}{\approx} \{B_t\}.$$

- 注3.5.3*. 固定N. 则$\{\tilde{S}_n = \sqrt{N}B_{\tilde{\sigma}_n}\}$ 是SRW, 其中, $\tilde{\sigma}_0 = 0$,

$$\tilde{\sigma}_n := \inf\{t \geqslant \sigma_{n-1} : \left|B_t - B_{\tilde{\sigma}_{n-1}}\right| = 1/\sqrt{N}.$$

- "$\{\tilde{S}_t^{(N)}\} \xrightarrow{P} \{B_t\}$" : $t = n/N$, $\tilde{\sigma}_n \approx nE\tilde{\sigma}_1 = n \times \frac{1}{N}$.

$$\tilde{S}_t^{(N)} \approx \tilde{S}_n/\sqrt{N} = B_{\tilde{\sigma}_n} \approx B_{n/N} = B_t.$$

- $\tilde{\sigma}_1 \overset{d}{=} \sigma_1/N$ (理同例3.4.5). 重点: $E\sigma_1 = 1.$

- 记$\tau = \tau_a \wedge \tau_b$.
- 命题3.5.6 (Wald第二引理).

$$E_x(B_\tau - x)^2 = E_x\tau.$$

- 注: 证明超过课程难度范围.
- 推论3.5.7. $E_x\tau = (x-a)(b-x)$.
- 证: $\star = \star = \frac{x-a}{b-a}(b-x)^2 + \frac{b-x}{b-a}(a-x)^2 = (x-a)(b-x)$.
- 引理3.5.5. $\sup_{a\leqslant x\leqslant b} E_x\tau^\alpha < \infty, \forall \alpha > 0$.
- 证: 略. (注: 证明在课程难度范围.)

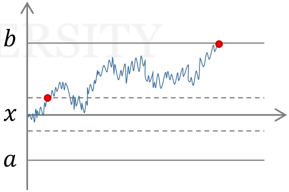
狄利克雷问题、泊松问题

- 区域$D = [a,b] \subseteq \mathbb{R}$. 边界: a, b. $\tau = \tau_{\partial D}$.
- $x \in D$,

$$P_x(\tau_b < \tau_a) = P_x(B_\tau = b) = E_x \mathbf{1}_{\{B_\tau = b\}}.$$

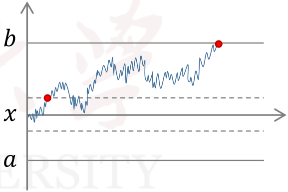
- $\varphi(x) = E_x f(B_\tau)$. (例: $f(x) = \mathbf{1}_{\{x=b\}}$.)
- 则$\varphi(x) = \frac{1}{2}\varphi(x+\delta) + \frac{1}{2}\varphi(x-\delta)$.
- 命题3.5.8(狄利克雷问题).

$$\begin{cases} \varphi''(x) = 0, & \forall x \in (a,b), \\ \lim\limits_{y\in(a,b), y\to x} \varphi(y) = f(x), & x = a \text{ 或 } b. \end{cases}$$



- $E_x\tau = E_x\int_0^\tau \mathbf{1}\mathrm{d}t$.
- $\psi(x) := E_x\int_0^\tau g(B_t)\mathrm{d}t$. (例: $g(x)\equiv 1$.)
- 则$\psi(x) = E_x\int_0^{\sigma_\delta} g(B_t)\mathrm{d}t + \frac{1}{2}(\psi(x+\delta)+\psi(x-\delta))$,

  其中, $\sigma_\delta = \inf\{t : |B_t - x| = \delta\}$.



- $\psi(x+\delta)+\psi(x-\delta)-2\psi(x)$

  $\approx \; -2g(x)\times E_x\sigma_\delta = \; -2g(x)\delta^2$.
- 命题3.5.8 (泊松问题). 若g 有界连续, 则

$$\begin{cases} \psi''(x) = -2g(x), & \forall x\in(a,b), \\ \lim\limits_{y\in(a,b),y\to x}\psi(y) = 0, & \forall x\in\{a,b\}. \end{cases}$$

二、高维情形及其应用

- $D \subseteq \mathbb{R}^d$, 为“好区域”, $\tau = \tau_{\partial D}$.
- $f : \partial D \to \mathbb{R}$ 连续; $g : D \to \mathbb{R}$ 有界连续.

$$\varphi(\vec{x}) := E_{\vec{x}} f\left(\vec{B}_\tau\right), \quad \psi(\vec{x}) := E_{\vec{x}} \int_0^\tau g(\vec{B}_t)\mathrm{d}t.$$

- 命题3.5.11. 记Laplace 算子$\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. 则

$$\begin{cases} \Delta\varphi(\vec{x}) = 0, & \forall \vec{x} \in D, \\ \lim\limits_{\vec{y}\in D, \vec{y}\to\vec{x}} \varphi(\vec{y}) = f(\vec{x}), & \forall \vec{x} \in \partial D. \end{cases}$$

$$\begin{cases} \Delta\psi(\vec{x}) = -2g(\vec{x}), & \forall \vec{x} \in D, \\ \lim\limits_{\vec{y}\in D, \vec{y}\to\vec{x}} \psi(\vec{y}) = 0, & \forall \vec{x} \in \partial D. \end{cases}$$

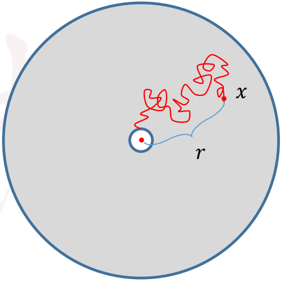
- 注: 连续可减弱为分段连续.

例3.5.14. d 为BM 的常返性.

- $\tau_r = \inf\{t \geqslant 0 : \|\vec{B}_t\| = r\}$.
- $D = \{\vec{x} \in \mathbb{R}^d : \varepsilon \leqslant \|\vec{x}\| \leqslant R\}$. 求: $P_{\vec{x}}(\tau_\varepsilon < \tau_R)$.
- 注: $P_0(\tau_R < \infty) = 1$;

  $\uparrow \lim\limits_{R\to\infty} \tau_R = \infty$, a.s..

  故$P_{\vec{x}}(\tau_\varepsilon < \infty) = \lim\limits_{R\to\infty} P_{\vec{x}}(\tau_\varepsilon < \tau_R)$.



- $\tau = \tau_{\partial D}$, $\varphi(\vec{x}) = E_{\vec{x}} f(\vec{B}_\tau)$.

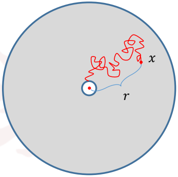
$$f(\vec{y}) = \begin{cases} 1, & 若\|\vec{y}\| = \varepsilon; \\ 0, & 若\|\vec{y}\| = R. \end{cases}$$

- 由各项同性: 令$\varphi(\vec{x}) = F(z)$, 其中 $z = r^2 = \|\vec{x}\|^2 = \sum_{i=1}^d x_i^2$.

- $P_{\vec{x}}(\tau_\varepsilon < \tau_R) \triangleq \varphi(\vec{x}) = F(z),\ z = r^2 = \|\vec{x}\|^2 = \sum_{i=1}^d x_i^2.$
- $\Delta\varphi(\vec{x}) = 0,\ \vec{x} \in D;\quad \varphi|_{\partial D} = f.$ 令$G = F'$,

$$4zF''(r^2) + 2dF'(r^2) = 0, \quad \text{i.e.} \quad 2zG'(z) + dG(z) = 0.$$

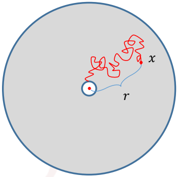
- $(\ln G(s))' = -\frac{d}{2} \cdot \frac{1}{z} \Rightarrow G(z) = C \cdot z^{-d/2}.$
- 由方程& 边界条件可解得$\varphi(\vec{x})$.
- $d = 1.\ r < x < R,\ \varphi(x) = (R - r)/(R - \varepsilon).$
- 注: $P_x(\tau_0 < \infty) = 1$, 点常返.
- $d = 2.\ \varphi(\vec{x}) = (\ln R - \ln r)/(\ln R - \ln \varepsilon).$
- 注: $P_x(\tau_\varepsilon < \infty) = 1$, 但$P_x(\tau_0 < \infty) = 0$, 集合常返.



$$P_{\vec{x}}(\tau_\varepsilon < \infty) = \lim_{R\to\infty} P_{\vec{x}}(\tau_\varepsilon < \tau_R) = 1;$$

$$P_x(\tau_0 < \tau_R) \leqslant P_{\vec{x}}(\tau_\varepsilon < \tau_R) \xrightarrow{\varepsilon\to 0} 0; \quad \star \xrightarrow{R\to\infty} P_x(\tau_0 < \infty).$$

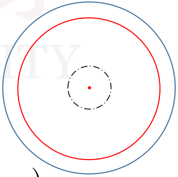
- $d \geqslant 3$. $\varphi(\vec{x}) = (\frac{1}{r^{d-2}} - \frac{1}{R^{d-2}})/(\frac{1}{\varepsilon^{d-2}} - \frac{1}{R^{d-2}})$.
- 注：非常返.



$$P_x(\tau_\varepsilon < \infty) = \lim_{R\to\infty} P_x(\tau_\varepsilon < \tau_R) = (\frac{\varepsilon}{r})^{d-2} < 1.$$

- §3.5 习题7. $d \geqslant 3$, 证明$\|\vec{B}_t\| \to \infty$, a.s..
- 证：固定k. 考虑τ_n,

$$A_n = \{\exists t \in [\tau_n, \tau_{n+1}] \text{ 使得} \|\vec{B}_t\| = k\}.$$



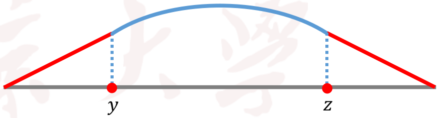
- 取$r = n$, $R = n+1$, $\varepsilon = k$. 当$n \gg k$ 时,

$$P(A_n) = (\frac{1}{n^{d-2}} - \frac{1}{(n+1)^{d-2}})/(\frac{1}{k^{d-2}} - \frac{1}{(n+1)^{d-2}})$$
$$\leqslant \frac{1}{n^{d-2}} / \frac{1}{2k^{d-2}}.$$

例3.5.16. $D=[0,1]$, $\tau=\tau_{\partial D}$. 给定$0\leqslant y<z\leqslant 1$, 求

$$\psi(x):=E_x\int_0^{\tau}\mathbf{1}_{\{B_t\in(y,z)\}}\mathrm{d}t.$$

- $\psi''(x)=-2,\ \forall x\in(y,z)$;　$\psi''(x)=0$, 其他.
  $\psi(0)=\psi(1)=0$.
- 令$h(x)=-x^2+ax+b$.
- $\psi'(y)=-2y+a=\frac{h(y)}{y}$,
  $\psi'(z)=-2z+a=-\frac{h(z)}{1-z}$.



- $a=2z-(z^2-y^2)$, $b=-y^2$. 解得ψ.
- $\psi(x)=\int_y^z p(x,w)\mathrm{d}w$,

$$p(x,w)=\begin{cases}2x(1-w), & 若\ 0\leqslant x\leqslant w\leqslant 1;\\ 2w(1-x), & 若\ 0\leqslant w<x\leqslant 1.\end{cases}$$

- 注: 例3.5.17*(略), 对比例1.7.7.

§3.6 布朗桥与O-U 过程

马氏过程.

- 定义3.5.1. $\{X_t, t \geqslant 0\}$ 满足: $\forall 0 \leqslant t_1 < \cdots < t_n < t$, $s > 0$, $x \in \mathbb{R}$, $D_1, \cdots, D_n, D \in \mathcal{B}(\mathbb{R})$,

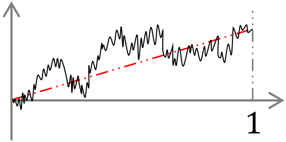
$$P(X_{t+s} \in D | X_t = x, X_{t_1} \in D_1, \cdots, X_{t_n} \in D_n) = p_s(x, D).$$

- 例3.5.4. 反射布朗运动. $X_t = |B_t|$,

$$p_s(x, y) = p_s(x, y) + p_s(x, -y), \quad \forall x, y \geqslant 0.$$

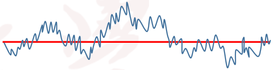
- 例3.5.5. 常系数扩散过程. $X_t = \sigma B_t + \mu t$.

一、布朗桥



- 背景: $\color{blue}{X_t} = B_t - \color{red}{tB_1}$, $0 \leqslant t \leqslant 1$.

- 刻画:
  (BB1) 轨道连续;
  (BB2) 高斯过程;
  (BB3.1) $EX_t = 0$;
  (BB3.2) $\forall s \leqslant t$, $EX_sX_t = s(1-t)$.
  (LHS $= E(B_s - sB_1)(B_t - tB_1) =$ RHS.)



- 定义3.6.1. 满足(BB1), (BB2), (BB3.1) & (BB3.2) 的$\{X_t : 0 \leqslant t \leqslant 1\}$ 为布朗桥.

- 注: $2s(1-t)$ 见例3.5.16. 特别地, $EX_t^2 = t(1-t)$.

- $\{X_t : 0 \leqslant t \leqslant 1\}$ 与B_1 独立:

$$EX_tB_1 = E(B_t - tB_1)B_1 = EB_tB_1 - tEB_1^2 = 0.$$

- 等价刻画: 当$B_1 = 0$ 时, $X_t = B_t$. 故,

$$\begin{aligned}\mathcal{L}(\{X_t : 0 \leqslant t \leqslant 1\}) =& \mathcal{L}(\{X_t : 0 \leqslant t \leqslant 1\}|B_1 = 0) \\ =& \mathcal{L}(\{B_t : 0 \leqslant t \leqslant 1\}|B_1 = 0),\end{aligned}$$

- 联合密度为:

$$p_{X_{t_1},\ldots,X_{t_n}}(x_1, \ldots, x_n) = p_{B_{t_1},\ldots,B_{t_n}|B_1}(x_1, \ldots, x_n|0).$$

例*: 经验分布$F_n(x) = \frac{1}{n}\sum_{m=1}^{n} 1_{\{X_m \leqslant x\}}$, $F(x) = P(X \leqslant x)$.

- 设X 的密度严格正.
- SLLN: $P(F_n(x) \to F(x)) = 1, \forall x$.
- CLT: $D_n(x) = \sqrt{n}(F_n(x) - F(x)) \xrightarrow{d} D(x)$,

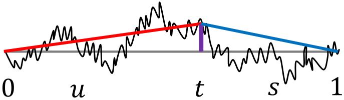
  $$D(x) \sim N(0, \sigma_x^2), \quad \sigma_x^2 = F(x)(1 - F(x)).$$
- "IP": $\{D_n(x) : x \in \mathbb{R}\} \xrightarrow{d} \{D(x) \triangleq D_x : x \in \mathbb{R}\}$.

  $ED_xD_y = \lim_n ED_n(x)D_n(y) = F(x)(1 - F(y)), x \leqslant y$.
- $F : \mathbb{R} \to (0, 1)$, 记$t = F(x)$.

  将D_x 改写为X_t, 即$X_t := D_{F^{-1}(t)}$.
- 视$t = F(x) < s = F(y)$, 则$EX_tX_s = t(1 - s)$, $t \leqslant s$.
- $\{X_t\}$ 是布朗桥.
- Donsker定理: $\max_{x\in\mathbb{R}} |D_n(x)| \xrightarrow{d} \max_{0\leqslant t\leqslant 1} |X_t|$.

- $Y_u = X_u - \frac{u}{t}X_t,\ 0 \leqslant r \leqslant t.$

  $Z_s = X_s - \frac{1-s}{1-t}X_t,\ t \leqslant s \leqslant 1.$



- $X_t, \{Y_u : 0 \leqslant u \leqslant t\}, \{Z_s : t \leqslant s \leqslant 1\}$ 相互独立:

$$EY_uX_s = EX_uX_s - \frac{u}{t}EX_tX_s = u(1-s) - \frac{u}{t}t(1-s) = 0,$$

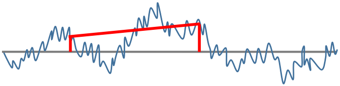
$$\text{故 } EY_uZ_s = 0. \quad \text{类似地}, EX_tZ_s = 0.$$

- $\{Y_u\}$: 长度为t 的布朗桥(定义3.6.2).

  (BB1), (BB2), (BB3.1) & (BB3.3)

  (BB3.3) $\forall 0 \leqslant u \leqslant v \leqslant t,\ EY_uY_v = u(t-v)/t.$

- §3.6 习题5.



二、Ornstein-Uhlenbeck(O-U)过程

- 设$\alpha \neq 0$. $X_t = e^{-\alpha t} B_{e^{2\alpha t}}, \ \forall t \in \mathbb{R}$.
- $X_{s+t} = e^{-\alpha(t+s)} B_{e^{2\alpha(t+s)}}$.
- $B_{e^{2\alpha(t+s)}} = B_{e^{2\alpha s}} \bigoplus \left(B_{e^{2\alpha(s+t)}} - B_{e^{2\alpha s}}\right)$.
- $$\begin{aligned} X_{s+t} &= e^{-\alpha t} \cdot e^{-\alpha s} B_{e^{2\alpha s}} \bigoplus e^{-\alpha(s+t)} \left(B_{e^{2\alpha(s+t)}} - B_{e^{2\alpha s}}\right) \\ &= e^{-\alpha t} X_s + \sigma Z. \end{aligned}$$
- $\sigma^2 = e^{-2\alpha(s+t)} (e^{2\alpha(s+t)} - e^{2\alpha s}) = 1 - e^{-2\alpha t}$.
- 转移密度:

$$q_t(x,y) = \frac{1}{\sqrt{2\pi(1-e^{-2\alpha t})}} \exp\left\{-\frac{(y-e^{-\alpha t}x)^2}{2(1-e^{-2\alpha t})}\right\}.$$

- 不变分布: $X_t \sim N(0,1), \ \forall t$.

- $X_t = e^{-\alpha t} B_{e^{2\alpha t}} \sim N(0,1)$,

$$\tilde{p}_t(x,y) = \frac{1}{\sqrt{2\pi(1-e^{-2\alpha t})}} \exp\left\{-\frac{(y-e^{-\alpha t}x)^2}{2(1-e^{-2\alpha t})}\right\}.$$

- 不变分布、强遍历、细致平衡条件:

$$\int \phi(x)\cdot q_t(x,y)\mathrm{d}x = \phi(y), \quad \phi(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}};$$

$$q_t(x,y) \xrightarrow{t\to\infty} \phi(y);$$

$$\phi(x)\cdot q_t(x,y) = \phi(y)\cdot q_t(y,x).$$

- 例3.6.5. 可逆过程. 令$Y_t := X_{-t}$, 则$\{Y_t\} \stackrel{d}{=} \{X_t\}$.
- 令$W_s = sB_{1/s}$, 则

$$Y_t = e^{\alpha t}\cdot B_{1/e^{2\alpha t}} = e^{\alpha t}\cdot W_{e^{2\alpha t}}/e^{2\alpha t} = e^{-\alpha t} W_{e^{2\alpha t}}.$$

§3.7 随机积分与随机微分方程简介

- 目标: 对随机过程$\{f_t\}$ 定义$\int_0^t f_u \mathrm{d}B_u$.
- 困难: 不能先固定ω, 因为 $\mathrm{d}B_u(\omega) = B'_u(\omega)\mathrm{d}u$ 不存在.
- 直观: $\int_0^t f_u \mathrm{d}B_u = \lim\limits_{|\Delta|\to 0} \sum_{i=0}^{n-1} f_{t_i} \cdot (B_{t_{i+1}} - B_{t_i})$.
  $\Delta : 0 = t_0 < t_1 < \cdots < t_n = t$, $|\Delta| = \max\limits_i (t_{i+1} - t_i)$.
- 解决: 不是 a.s. 收敛, 而是L^2 收敛: $E(X_n - X)^2 \to 0$.
- $\{f_t\}$ 满足若干要求:

  $f_t \in \mathscr{F}_t := \sigma(B|_{[0,t]})$, 与$\{B_{t+s} - B_s\}$ 独立; $\cdots\cdots$

- 结论: $\exists$ 轨道连续的$\{X_t\}$, 使得$\star\star \xrightarrow{L^2} X_t$, $\forall t$.
- 随机积分/伊藤积分: 记为$\int_0^t f_u \mathrm{d}B_u$.

- 积分的线性: $\int_0^t (c\cdot f_u + g_u)\mathrm{d}B_u = c\int_0^t f_u \mathrm{d}B_u + \int_0^t g_u \mathrm{d}B_u$.
- 期望: $E\int_0^t f_u \mathrm{d}B_u = 0$.
- 协方差: $E\int_0^t f_u \mathrm{d}B_u \cdot \int_0^t g_u \mathrm{d}B_u = \int_0^t Ef_u g_u \mathrm{d}u$.
  - 记$\Delta_i := B_{t_{i+1}} - B_{t_i}$.
  - 若$i<j$, 此时$t_i < t_{i+1} \leqslant t_j < t_{j+1}$,
    则$f_{t_i}, \Delta_i, g_{t_j} \in \mathscr{F}_{t_j}$, 与 Δ_j 独立. 故

$$Ef_{t_i}\Delta_i g_{t_j}\Delta_j = 0, \quad \text{同理 } Ef_{t_i}\Delta_i g_{t_j}\Delta_j = 0, \ \forall i > j.$$

  - 若$i=j$, 则f_{t_i}, g_{t_i} 与Δ_i 独立. 故

$$Ef_{t_i}\Delta_i g_{t_i}\Delta_i = (Ef_{t_i}g_{t_i})E\Delta_i^2 = (Ef_{t_i}g_{t_i})\Delta t_i$$

- 特别地, $E\int_0^s f_u \mathrm{d}B_u \cdot \int_s^t g_u \mathrm{d}B_u = 0$, $\forall s < t$.

例3.7.3. 布朗轨道的时间变换.

- $f:[0,\infty)\to(0,\infty)$, 实函数/非随机. $f_t(\omega):=f(t)$
- $\{X_t=\int_0^t f_u \mathrm{d}B_u\}$ 是高斯过程, 轨道连续, $EX_t=0$.
- $\forall t\leqslant s$,
$$EX_tX_s=\int_0^t f_u^2\mathrm{d}u \triangleq \varphi(t).$$
- 记$\varphi^{-1}=\psi$. 令$Y_t=X_{\psi(t)}$, $t\geqslant 0$.
- $\{Y_t\}$ 是布朗运动: $\forall t\leqslant s$, $\psi(t)\leqslant\psi(s)$,
$$EY_tY_s=EX_{\psi(s)}X_{\psi(t)}=\varphi(\psi(t))=t.$$
- $X_t=Y_{\varphi(t)}$.

例3.7.4: 求$\int_0^t B_u \mathrm{d}B_u$.

- $\Delta := 0 = t_0 < \ldots < t_n = t, \quad |\Delta| = \max\limits_{0 \leqslant i \leqslant n-1} (t_{i+1} - t_i) \to 0.$
- $f_{t_i} = B_{t_i}$. 计算 $\lim_n \sum_i B_{t_i} \Delta_i$. $(\Delta_i = B_{t_{i+1}} - B_{t_i})$
- $B_{t_{i+1}} = B_{t_i} + \Delta_i$, 故

$$2B_{t_i}\Delta_i = B_{t_{i+1}}^2 - B_{t_i}^2 - \Delta_i^2.$$

- $\sum_i \Delta_i^2 \xrightarrow{L^2} t$:
  - $E\Delta_i^2 = t_{i+1} - t_i \Rightarrow E\sum_i \Delta_i^2 = t$.
  - $E\left(\sum_i \Delta_i^2 - t\right)^2 = \mathrm{Var}(\sum_i \Delta_i^2) = \sum_i \mathrm{Var}(\Delta_i^2)$

    $= \sum_i \mathrm{Var}\left((t_{i+1} - t_i)Z^2\right) = \sum_i (t_{i+1} - t_i)^2 \ \mathrm{Var}(Z^2) \to 0.$
- 解为$\frac{1}{2}(B_t^2 - t)$.

随机微分(积分)方程.

- 积分方程: $X_t = X_0 + \int_0^t \sigma_u \mathrm{d}B_u + \int_0^t b_u \mathrm{d}u.$
- 例: $\int_0^t B_u \mathrm{d}B_u = \frac{1}{2}(B_t^2 - t)$. 即, $B_t^2 = \int_0^t 2B_u \mathrm{d}B_u + \int_0^t 1 \mathrm{d}u.$
- 微分方程: $dX_t = \sigma_t \mathrm{d}B_t + b_t \mathrm{d}t.$
- 例: $\mathrm{d}B_t^2 = 2B_t \mathrm{d}B_t + 1\mathrm{d}t.$
- $\mathrm{d}\varphi(B_t) = \varphi'(B_t)\mathrm{d}B_t + \frac{1}{2}\varphi''(B_t)\mathrm{d}t.$ (例, $\varphi(x) = x^2$.)
- $\varphi(B_t + \mathrm{d}B_t) - \varphi(B_t) = \varphi'(B_t)\mathrm{d}B_t + \frac{1}{2}\varphi''(B_t)\mathrm{d}B_t^2 + o(\mathrm{d}B_t^2),$
  $\mathrm{d}B_t = B_{t+\mathrm{d}t} - B_t,$ $(\mathrm{d}B_t)^2 = \mathrm{d}t.$ $(\mathrm{d}X_t)^2 = \sigma_t^2 \mathrm{d}t.$
- Itô 公式: 令$Y_t = \varphi(t, X_t)$, 则

$$\mathrm{d}Y_t = \frac{\partial \varphi}{\partial x}(t, X_t)\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2}(t, X_t)(\mathrm{d}X_t)^2 + \frac{\partial \varphi}{\partial t}(t, X_t)\mathrm{d}t.$$

例3.7.6. 求解随机微分方程$\mathrm{d}X_t = \sigma \mathrm{d}B_t - bX_t\mathrm{d}t$.

- 注: $X_t = X_0 + \sigma B_t - \int_0^t bX_u \mathrm{d}u$. 这不是解.
- $\mathrm{d}X_t + bX_t\mathrm{d}t = \sigma \mathrm{d}B_t$.
- 令$\varphi(t,x) = e^{bt}x$, $Y_t = \varphi(t, X_t)$. 则

$$\mathrm{d}Y_t = e^{bt}(\mathrm{d}X_t + bX_t\mathrm{d}t) = e^{bt}\sigma \mathrm{d}B_t.$$

- $Y_t = Y_0 + \sigma \int_0^t e^{bs}\mathrm{d}B_s$. 其中, $Y_0 = \varphi(0, X_0) = X_0$.
- $X_t = e^{-bt}Y_t = e^{-bt}X_0 + \sigma e^{-bt}\int_0^t e^{bu}\mathrm{d}B_u$.
- 取$\alpha > 0$, $\sigma = \sqrt{2\alpha}$, $b = \alpha$.
- 假设$X_0 \sim N(0,1)$, 与$\{B_t\}$ 独立. 则$\{X_t\}$ 是O-U 过程.

例3.7.7. Black-Scholes 模型. $\mathrm{d}X_t = X_t(\sigma \mathrm{d}B_t + \mu \mathrm{d}t)$.

- σ: 波动率, μ: 平均收益率.
- $\mathrm{d}X_t - X_t(\sigma \mathrm{d}B_t + \mu \mathrm{d}t) = 0$.
- 取$\varphi(x) = \ln x$, 令$Y_t = \varphi(X_t)$. 则

$$\begin{aligned}\mathrm{d}Y_t =& \frac{1}{X_t}\mathrm{d}X_t - \frac{1}{2}\cdot\frac{1}{X_t^2}(\mathrm{d}X_t)^2 \\ =&(\sigma \mathrm{d}B_t + \mu \mathrm{d}t) - \frac{1}{2}\cdot\frac{1}{X_t^2}(\sigma X_t)^2 \mathrm{d}t \\ =&\sigma \mathrm{d}B_t + (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t.\end{aligned}$$

- $Y_t = Y_0 + \sigma B_t + (\mu - \frac{1}{2}\sigma^2)t$. 其中, $Y_0 = \ln X_0$.
- 故, $X_t = X_0 \exp\{\sigma B_t + (\mu - \frac{1}{2}\sigma^2)t\}$.