§3.4 两个随机变量的函数

- 设$\xi=(X,Y)$ 有联合密度$p(x,y)$, 求$Z=f(X,Y)$ 的密度.
- 分布函数法: 第一步, 用$p(x,y)$ 表达F_Z,

$$F_Z(z)=P(f(X,Y)\leqslant z)=\iint\limits_{\{(x,y):f(x,y)\leqslant z\}} p(x,y)dxdy.$$

- 第二步, 将$\star\star$ 化为如下积分,

$$\star\star=\int_{-\infty}^{z} p(u)du.$$

- 结论: $p_Z(z)=p(z)$.

例7.3. 假设X 与Y 相互独立, $X \sim \mathcal{P}(\lambda_1)$, $Y \sim \mathcal{P}(\lambda_2)$.
令$Z = X + Y$. 求: Z 的分布; 在$Z = n$ 的条件下, X 的条件分布.

- $Z \sim \mathcal{P}(\lambda_1 + \lambda_2)$: $\forall n \geqslant 0$, $P(Z = n)$

$$= \sum_{k=0}^{n} P(X = k, Y = n - k) = \sum_{k=0}^{n} \frac{\lambda_1^k}{k!} e^{-\lambda_1} \frac{\lambda_2^{n-k}}{(n-k)!} e^{-\lambda_2}$$

$$= \frac{1}{n!} \left(\sum_{k=0}^{n} C_n^k \lambda_1^k \lambda_2^{n-k} \right) e^{-(\lambda_1+\lambda_2)} = \frac{(\lambda_1 + \lambda_2)^n}{n!} e^{-(\lambda_1+\lambda_2)}.$$

- $n = 0$ 时, $P(X = 0|Z = 0) = 1$.
- $n > 0$ 时, 记$p = \frac{\lambda_1}{\lambda_1+\lambda_2}$, $q = 1 - p$. 则, $k = 0, 1, \cdots n$,

$$P(X = k|Z = n) = \frac{C_n^k \lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} = C_n^k p^k q^{n-k},$$

- 条件分布是二项分布.

例7.4. 假设X 与Y 相互独立, $X \sim B(n_1, p)$, $Y \sim B(n_2, p)$, $0 < p < 1$. 令$Z = X + Y$. 求: Z 的分布; 在$Z = n$ 的条件下, X 的条件分布.

- $n = 0, 1 \cdots, n_1 + n_2$. 记$q = 1 - p$, 则$P(Z = n)$

$$= \sum_{k=0}^{n} P(X = k, Y = n - k) = \sum_{k=0}^{n} C_{n_1}^{k} C_{n_2}^{n-k} p^{k+n-k} q^{n_1-k+n_2-(n-k)}$$

$$= \sum_{k=0}^{n} C_{n_1}^{k} C_{n_2}^{n-k} p^{n} q^{n_1+n_2-n} = C_{n_1+n_2}^{n} p^{n} q^{n_1+n_2-n}.$$

- $n = 0$ 时, $P(X = 0|Z = 0) = 1$.
- $n > 0$ 时, $k = 0, 1, \cdots n$,

$$P(X = k|Z = n) = \frac{C_{n_1}^{k} C_{n_2}^{n-k}}{C_{n_1+n_2}^{n}},$$

- 条件分布是超几何分布.

- 定理4.1. 设$\xi = (X, Y)$ 有联合密度$p(x, y)$, $Z = X + Y$. 则

$$p_Z(z) = \int_{-\infty}^{\infty} p(x, z - x)dx.$$

- 证: 第一步,

$$F_Z(z) = P(X + Y \leqslant z) = \iint_{x+y\leqslant z} p(x, y)dxdy.$$

- 第二步,

$$\star = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} p(x, y)dy \right) dx = \int_{-\infty}^{z} \int_{-\infty}^{\infty} p(x, u - x)dxdu$$

- 推论(系4.1): 若X, Y 相互独立, 分别有密度p_X, p_Y, 则$Z = X + Y$ 是连续型, 且

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)p_Y(z - x)dx = \int_{-\infty}^{\infty} p_X(z - y)p_Y(y)dy.$$

例4.1 & 4.2. 设(X,Y) 服从二维正态分布, 联合密度$p(x,y)$ 为

$$p(x,y)=\hat{C}\exp\left\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right\},\quad \left(u=\frac{x-\mu_1}{\sigma_1},v=\frac{y-\mu_2}{\sigma_2}\right),$$

其中, $\hat{C}=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$. 求$Z=X+Y$ 的密度.

- $p_Z(z)=\int_{-\infty}^{\infty}p(x,z-x)dx$. 当$y$ 取$z-x$ 时,

$$v=\frac{y-\mu_2}{\sigma_2}=\frac{z-(\mu_1+\sigma_1 u)-\mu_2}{\sigma_2}=C-\frac{\sigma_1}{\sigma_2}u,$$

  其中, $C=(z-\mu_1-\mu_2)/\sigma_2$.

- 此时, $u^2-2\rho uv+v^2$

$$\begin{aligned}&=u^2-2\rho u\left(C-\frac{\sigma_1 u}{\sigma_2}\right)+\left(C-\frac{\sigma_1 u}{\sigma_2}\right)^2\\&=\left(1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2\right)u^2-2\left(\rho+\frac{\sigma_1}{\sigma_2}\right)Cu+C^2.\end{aligned}$$

- 目标: 计算$p_Z(z)=\int_{-\infty}^{\infty}p(x,z-x)dx$. 已有:

$$p(x,z-x)=\hat{C}\left\{-\frac{Au^2-2Bu+C^2}{2(1-\rho^2)}\right\},\quad 其中, u=\frac{x-\mu_1}{\sigma_1},$$

$$A=1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2,\quad B=\left(\rho+\frac{\sigma_1}{\sigma_2}\right)C,\quad C=\frac{z-(\mu_1+\mu_2)}{\sigma_2}.$$

- 配方:

$$Au^2-2Bu+C^2=A\left(u-\frac{B}{A}\right)^2-\left(\frac{B^2}{A}-C^2\right).$$

- 于是, $p_Z(z)$

$$=\hat{C}\exp\left\{\frac{\frac{B^2}{A}-C^2}{2(1-\rho^2)}\right\}\times\int_{-\infty}^{\infty}\exp\left\{-\frac{A\left(u-\frac{B}{A}\right)^2}{2(1-\rho^2)}\right\}\sigma_1du$$

$$=\tilde{C}\exp\left\{\frac{B^2-AC^2}{2(1-\rho^2)A}\right\}.\quad \tilde{C}=\hat{C}\sigma_1\sqrt{2\pi\frac{1-\rho^2}{A}}=\frac{1}{\sqrt{2\pi\sigma_2^2A}}.$$

- 已有: $p_Z(z) = \tilde{C}\exp\left\{\frac{B^2-AC^2}{2(1-\rho^2)A}\right\}$, 其中$\tilde{C}$ 是常数,

$$A = 1+2\rho\frac{\sigma_1}{\sigma_2}+\left(\frac{\sigma_1}{\sigma_2}\right)^2,\ B = \left(\rho+\frac{\sigma_1}{\sigma_2}\right)C,\ C = \frac{z-(\mu_1+\mu_2)}{\sigma_2}.$$

- $B^2 - AC^2$

$$= \left(\left(\rho+\frac{\sigma_1}{\sigma_2}\right)^2 - A\right)C^2 = (\rho^2-1)\frac{(z-(\mu_1+\mu_2))^2}{\sigma_2^2}.$$

- 因此,

$$p_Z(z) = \tilde{C}\exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\} = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\}.$$

其中, $\mu = \mu_1+\mu_2$, $\sigma^2 = \sigma_2^2 A = \sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$.

- 特别地, 若$\rho = 0$ (即X, Y 相互独立), 则

$$X + Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

- 定理4.2. 设(X,Y) 有联合密度$p(x,y)$.
  令$Z=X/Y$ (当$Y=0$ 时, 规定$Z=0$). 则Z 为连续型, 且
$$p_Z(z)=\int_{-\infty}^{\infty}|y|p(zy,y)dy.$$
- 证明: 第一步, $\frac{x}{y}\leqslant z$ 当且仅当 "$y>0$ 且$x\leqslant yz$" 或者 "$y<0$ 且$x\geqslant yz$." 于是,
$$F_Z(z)=P(Y>0,X\leqslant Yz)+P(Y<0,X\geqslant Yz).$$
- $P(Y>0,X\leqslant Yz)$
$$=\int_0^{\infty}\int_{-\infty}^{yz}p(x,y)dxdy=\int_0^{\infty}\int_{-\infty}^{z}p(yu,y)ydudy$$
$$=\int_{-\infty}^{z}\left(\int_0^{\infty}yp(yu,y)dy\right)du.$$
- 类似处理★★, 即可.

例4.4. X, Y 相互独立, 都服从$N(0,1)$. 求$Z = X/Y$ 的密度.

- 联合密度:

$$p(x,y) = \frac{1}{2\pi} \exp\left\{-\frac{x^2+y^2}{2}\right\}.$$

- 因此,

$$p_Z(z) = \int_{-\infty}^{\infty} |y| p(zy, y) dy = \int_{-\infty}^{\infty} |y| \frac{1}{2\pi} \exp\left\{-\frac{(zy)^2+y^2}{2}\right\} dy$$

$$= \frac{2}{2\pi} \int_0^{\infty} y \exp\left\{-\frac{(z^2+1)y^2}{2}\right\} dy$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-(z^2+1)u} du = \frac{1}{\pi(z^2+1)}.$$

- 定理4.3. 假设$\xi = (X, Y)$ 为连续型, 有密度$p(x, y)$.

假设

$$\eta = (U, V), \qquad \text{其中 } U = f(X, Y), \quad V = g(X, Y).$$

如果(1) $P(\xi \in A) = 1$ 且$(f, g) : A \to G$ 是一对一的;

(2) $f, g \in C^1(A)$, 且$\frac{\partial(u,v)}{\partial(x,y)} \neq 0$, $\forall (x, y) \in A$,

那么, η 是连续型, 且

$$p_{U,V}(u, v) = p\Big(x(u, v), y(u, v)\Big)\left|\frac{\partial(x, y)}{\partial(u, v)}\right|, \quad (u, v) \in G.$$

- 证: $\forall D \subseteq G$, 找$D^* \subseteq A$ 使得$\eta \in D$ iff $\xi \in D^*$. 于是,

$$P(\star) = P(\star) = \iint_{D*} p(x, y)dxdy = \iint_{D} \star\star dudv.$$

例4.5, 4.7, & 习题三、21. 假设X, Y 相互独立, 都服从$N(0,1)$.

- 用极坐标表达:

$$X = R\cos\Theta, \quad Y = R\sin\Theta.$$

- $A = \{(x,y) : x \neq 0, y \neq 0\}$,

  $G = \{(r,\theta) : r > 0, 0 < \theta < 2\pi;\ \theta \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$.

- $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$

- $p_{R,\Theta}(r,\theta) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} |\frac{\partial(x,y)}{\partial(r,\theta)}|$

  $= \frac{1}{2\pi} r e^{-\frac{1}{2}r^2},\ r > 0,\ 0 < \theta < 2\pi;\ \sout{\theta \neq \frac{\pi}{2}, \pi, \frac{3\pi}{2}}.$

- R, Θ 独立: $p_{R,\Theta}(r,\theta) = p_R(r) \cdot p_\Theta(\theta)$.

- $W := R^2 = X^2 + Y^2 \sim \mathrm{Exp}(\frac{1}{2})$. 因为, $\forall w > 0$,

$$p_W(w) = p_R(r)\frac{dr}{dw} = r\exp\left\{-\frac{r^2}{2}\right\}\cdot\frac{1}{2r} = \frac{1}{2}e^{-\frac{w}{2}}.$$

- $U := e^{-\frac{1}{2}W} \sim U(0,1)$. 因为, $\forall p \in (0,1)$,

$$P(U \leqslant p) = P(W \geqslant -2\ln p) = e^{-\frac{1}{2}(-2\ln p)} = e^{\ln p} = p.$$

- $V := \frac{1}{2\pi}\Theta \sim U(0,1)$, 且$U$ 与V 相互独立.
- $R = \sqrt{-2\ln U}$, $\Theta = 2\pi V$, 即

$$X = \sqrt{-2\ln U}\cos(2\pi V), \quad Y = \sqrt{-2\ln U}\sin(2\pi V).$$

2.两个随机变量的函数的数学期望

- 随机向量函数的期望(定理4.6):

$$\text{离散型:} Ef(X,Y) = \sum_{i,j} f(x_i, y_j) P(X = x_i, Y = y_j).$$

$$\text{连续型:} Ef(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)p(x,y)dxdy.$$

- 定理4.4. 若X 与Y 相互独立, 则$EXY = (EX)\cdot(EY)$.
- 例, 连续型的证明:

$$EXY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyp_X(x)p_Y(y)dxdy = (EX)(EY).$$

- 定理4.5. 若X 与Y 相互独立,

  则$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$.
- 证: 左$= E\left(X + Y - (EX + EY)\right)^2$

  $= \text{右} + 2E(X - EX)(Y - EY)$.