Nullstellensatz in Chromatic Homotopy

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History

Chromatic homotopy theory was established by Quillen and Landweber who pointed out the following relationship between formal groups and complex-oriented spectra.

Theorem (Quillen-Landweber)

Every complex-oriented multiplicative ring spectrum admits a one-dimensional formal group over its homotopy ring. Conversely, given a commutative ring R with a flat formal group, we can construct a multiplicative even-graded complex-oriented spectrum whose π_0 is exactly R. In addition, the construction is functorial and fully faithful in $h\mathrm{Sp}$.

Hence, we could construct a lot of meaningful spectra to understand stable homotopy theory by virtue of various formal groups in each mathematical branch. For example, the formal completion of an abelian variety, as well as the identity component of a p-divisible group, is a formal group (maybe of higher dimension). The famous Lubin-Tate theory exactly comes from this case.

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To obtain a heuristic insight, people observed an invariant called height of a formal group (at a fixed prime p). Height theory can be regarded as strata of the moduli stack of formal groups. From the perspective of spectra, we believe that the height filtration in algebraic geometry mirrors nice properties in stable homotopy theory (e.g. Ravenel's Conjectures).

Explicit Figures

Figure: Traditional way to calculate stable homotopy groups

It corresponds to the diagram of Spec($\mathbb{Z}$) where each closed point means the ordinary p-localization.

Figure: Chromatic philosophy

Similarly, the diagram on the blackboard is the topological space of moduli stack of formal groups. It suggests the enhancement brought by spherical spectrum $\mathbb{S}$ compared with $\mathbb{Z}$ or $H\mathbb{Z}$. To our surprise, each block in the moduli stack has a homotopic explanation in Sp, i.e., Morava K-theory. As a result, we can execute a series of new localizing functors in a wider range, enlarging people's horizons.

Ravenel's Conjectures

Taking inspiration from algebraic geometry, Ravenel raised these renowned conjectures among which only the last one is wrong. I believe that these theorems are foundational outcomes of chromatic homotopy theory and they witnessed the first combination between homotopy and algebraic geometry.

1. Nilpotence theorem means all Morava K-theories form the prime fields in Sp.
2. Periodicity theorem implies that in each $K(n)$-local case, we will find many periods simplifying some calculations.
3. Chromatic convergence theorem tells us how to glue data at different heights together.
4. Telescope conjecture is people's bold attempt to calculate $K(n)$-localization. Actually, the corresponding $T(n)$-localization relates to $K(n)$-ones in a more subtle way.

Simultaneously, topologists like J. Peter May introduced a well-behaved notion of rings in homotopy theory called $\mathbb{E}_\infty$-rings. They showed that a majority of good objects are of this property and thus tried to develop a brave-new algebraic geometry based on these rings. So the Morava K-theory $K(n)$ means a much broader and more reasonable way of localization than only considering the classical ones.

Question

Is there a Nullstellensatz theorem in spectral algebraic geometry?

Classical Nullstellensatz Theorem

Recall the following famous theorem.

Theorem (Hilbert)

For an algebraically closed field k, every maximal ideal in the polynomial ring $k[x_1, \cdots, x_n]$ is of the form $(x_1 - a_1, \cdots, x_n - a_n)$ for each $a_i \in k$. Obviously, such a statement holds only if k is algebraically closed.

In order to match the setting of ∞-categories better, one can use this statement.

Restatement

Every residue field of $k[x_1, \cdots, x_n]$ admits a map to k if k is algebraically closed.

Main Theorem

Today I will show the chromatic version of Nullstellensatz theorem.

Definition

For a general ∞-category $\mathcal{C}$, we say an object $A \in \mathcal{C}$ satisfies Nullstellensatz theorem if in $\mathcal{C}_{/A}$ every nonfinal compact object admits a map to A.

Theorem (Nullstellensatz)

In $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, *a nontrivial ring R satisfies Nullstellensatz theorem if and only if it is equivalent to the Morava E-theory $E(L)$ for some L algebraically closed field.*

Theorem (Existence of Point)

For $0 \neq R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ *there is always a ring homomorphism* $R \to E(L)$ *where L is a algebraically closed field.*

Key Input

1. Spherical Witt vectors.
2. Lurie's enhancement of $\mathbb{E}_\infty$-Lubin-Tate theory.
3. Freeness and cofreeness of Lubin-Tate theory.

Sketch

First of all, we can assume R is an $E(k)$-algebra due to base change in which k is a perfect field of characteristic p. Then to prove the second theorem we will construct three maps and use Quillen's small object argument to prove the existence of points or, actually, Lubin-Tate cover.

- The natural map $f : E(k[t^{1/p^\infty}]) \to E(k[t^{\pm 1/p^\infty}]) \times E(k)$.
- A certain map $g : E(k)\{z^0\} \to E(k\{z^0\}^\sharp)$.
- $h : E(k)\{z^1\} \xrightarrow{z^1 \mapsto 0} E(k)$.

Proposition

Given a commutative $T(n)$-local $E(k)$-algebra R, we have:

1. $f \perp R$ (i.e. f has the right lifting property with respect to R) is equivalent to $R^\flat$ being of Krull dimension 0.
2. $g \perp R$ is equivalent to $\pi_0 E(R^\flat) \to \pi_0 R$ is subjective.
3. $h \perp R$ is equivalent to $\pi_1(R) = 0$.

If all of these points hold, then R is equivalent to the Lubin-Tate theory of a perfect k-algebra A of Krull dimension 0.

Remark

Using Quillen's small object argument, we could obtain the so-called Lubin-Tate cover. Note that Krull dimension 0 means that every ideal is a maximal ideal. So it is obvious that the Lubin-Tate cover is the disjoint union of each point with maybe different stalks. This idea makes us introduce the constructible spectral space of a ring spectrum.

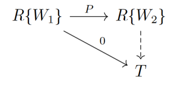
Sketch Continued

Thus we can prove the necessary part of the Nullstellensatz theorem, combining the existence of points and the following lemma.

Lemma

Let $\mathcal{C}$ be a presentable symmetric monoidal stable ∞-category. $R \in \mathrm{CAlg}(\mathcal{C})$ satisfies the Nullstellensatz theorem. Then for any compact R-module W_1 and W_2 and $P : R\{W_1\} \to R\{W_2\}$, the followings are equivalent.

1. *There exists a nonzero $T \in \mathrm{CAlg}_R(\mathcal{C})$ makes the following diagram commute.*



2. *Such T could be replaced by R itself.*

We often consider $R[x] \xrightarrow{x \mapsto P} R[y] \to R[y]/P$ in ordinary commutative algebra. If the last item is replaced by R, we have a root of polynomial P.

Redshift Conjecture

This conjecture is named after redshift because just as redshift in physics, algebraic K-theory makes the wavelength, or the height, increase by 1.

Conjecture

$\mathrm{ht}(K(R)) = \mathrm{ht}(R) + 1$, if R is a nontrivial commutative ring spectrum of height greater than 0.

Before 2022, mathematicians had shown that

Theorem (LMMT, Yuan)

In terms of the conjecture above, at least we know the left-hand side is less than the right-hand side. Furthermore, when R is equivalent to $E(L)$ for L algebraically closed field, the equality holds.

Proof of Redshift Conjecture.

One last thing we need to do is to show that the left-hand side is greater. Unwinding definition, it means that $T(n+1) \otimes K(R) \neq 0$ where n is the height of R. Otherwise, one can take a point $R \to E(L)$. The statement related to Morava E-theory implies that $T(n+1) \otimes K(E(L))$ is nontrivial. However, trivial algebra doesn't admit any nontrivial module or algebra. So the ring map $T(n+1) \otimes K(R) \to T(n+1) \otimes K(E(L))$ induces a contradiction. □