Optimal Dividend Policy of A Large Insurance Company with Solvency Constraints

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Outline

Methods for Making Maximal Profit · · ·

The insurance company generally takes the following means to earn maximal profit, reduce its risk exposure and improve its security:

- Proportional reinsurance

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- Controlling bankrupt probability(or solvency) and so on

Cramér-Lundberg model of cash flows

- The classical model with no reinsurance, dividend pay-outs

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The cash flow (reserve process) r_t of the insurance company follows

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where

claims arrive according to a Poisson process N_t with intensity ν on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

Cramér-Lundberg model of reserve process

U_i denotes the size of each claim. Random variables U_i are i.i.d. and independent of the Poisson process N_t with finite first and second moments given by μ_1 and μ_2.

$$p = (1+\eta)\nu\mu_1 = (1+\eta)\nu\mathbf{E}\{U_i\}$$

is the premium rate and $\eta > 0$ denotes the *safety loading*.

Diffusion approximation of Cramér-Lundberg model

By the central limit theorem, as $\nu \to \infty$,

$$r_t \overset{d}{\approx} r_0 + BM(\eta\nu\mu_1 t, \nu\mu_2 t).$$

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So we can assume that the cash flow $\{R_t, t \geq 0\}$ of insurance company is given by the following diffusion process

$$dR_t = \mu dt + \sigma dW_t,$$

where the first term " μt " is the income from insureds and the second term " σW_t " means the company's risk exposure at any time t.

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  The cash flow $\{R_t, t \geq 0\}$ of the insurance company then becomes

$$dR_t = (\mu - (1 - a(t))\lambda)dt + \sigma a(t)dW_t, \quad R_0 = x.$$

We generally assume that $\lambda \geq \mu$ based on real market.

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where $1 - a(t)$ is called the reinsurance fraction at time t, the $R_0 = x$ means that the initial capital is x, the constants μ and λ can be regarded as the safety loadings of the insurer and reinsurer, respectively.

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- A pair of $\mathcal{F}_t$ adapted processes $\pi = \{a_\pi(t), L_t^\pi\}$ is called a admissible policy if $0 \leq a_\pi(t) \leq 1$ and L_t^π is a nonnegative, non-decreasing, right-continuous with left limits.

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- Π denotes the whole set of admissible policies.
- When a admissible policy π is applied, the model (1) can be rewritten as follows:

$$dR_t^\pi = (\mu - (1 - a_\pi(t))\lambda)dt + \sigma a_\pi(t)dW_t - dL_t^\pi, \quad R_0^\pi = x. \tag{2}$$

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$$J(x,\pi) \;=\; \mathbb{E}\{\int_0^{\tau_x^\pi} e^{-ct} dL_t^\pi\} \tag{3}$$

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- Optimal control problem for the model (1) is to find the optimal return function $V(x)$ and the optimal policy π^* such that $V(x) = J(x,\pi^*)$

Solution of optimal control problem for the model (1) does not meet safety level

It well known that one can find a dividend level $b_0 > 0$, an optimal policy $\pi^*_{b_0}$ and an optimal return function $V(x, \pi^*_{b_0})$ to solve optimal control problem for the model (1), i.e.,

$$V(x) = V(x, b_0) = J(x, \pi^*_{b_0})$$

and b_0 satisfies

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However, the b_0 may be too low and it will make the company go bankrupt soon

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- Indeed, we proved that the b_0 and $\pi^*_{b_0}$ satisfy for any $0 < x \le b_0$ there exists $\varepsilon_0 > 0$ such that

$$\mathbf{P}\{\tau_x^{\pi^*_{b_0}} \le T\} \ge \varepsilon_0 > 0, \tag{5}$$

where

$$\varepsilon_0 = \min\Big\{\frac{4[1-\Phi(\frac{x}{d\sigma\sqrt{T}})]^2}{\exp\{\frac{2}{\sigma^2}(\lambda^2+\delta^2)T\}}, \frac{x}{\sqrt{2\pi}\sigma}\int_0^T t^{-\frac{3}{2}}\exp\{-\frac{(x+\mu t)^2}{2\sigma^2 t}\}dt\Big\},$$

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- If the company's preferred risk level is $\varepsilon (\leq \varepsilon_0)$, i.e.,

$$\mathbb{P}[\tau_x^{\pi^*_{b_0}} \leq T] \leq \varepsilon, \tag{6}$$

then the company has to reject the policy $\pi^*_{b_0}$ because it does not meet safety requirement (6) by (5), and the insurance company is a business affected with a public interest,

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We establish setting to solve the problems above as follows.

General setting optimal control problem for the model (1)with solvency constraints

- For a given admissible policy π the performance function

$$J(x,\pi) \quad = \quad \mathbb{E}\{ \int_0^{\tau_x^\pi} e^{-ct} dL_t^\pi \} \tag{7}$$

- The optimal return function

$$V(x) \quad = \quad \sup_{b \in \mathfrak{B}} \{ V(x,b) \} \tag{8}$$

- where $V(x,b) = \sup_{\pi \in \Pi_b} \{ J(x,\pi) \}$, solvency constraint set

  $\mathfrak{B} := \{ b \ : \ \mathbb{P}[\tau_b^{\pi_b} \leq T] \leq \varepsilon, J(x,\pi_b) = V(x,b)$ and $\pi_b \in \Pi_b \}$,

  $\Pi_b = \{ \pi \in \Pi : \int_0^\infty I_{\{s:R^\pi(s)<b\}} dL_s^\pi = 0 \}$ with property:
  $\Pi = \Pi_0$ and $b_1 > b_2 \Rightarrow \Pi_{b_1} \subset \Pi_{b_2}$.

Main goal

Finding value function $V(x)$, an optimal dividend policy $\pi^*_{b^*}$ and the optimal dividend level b^* to solve the sub-optimal control problem (7) and (8), i.e., $J(x, \pi^*_{b^*}) = V(x)$.

Our main results are the following

Main Results

Theorem

Assume that transaction cost $\lambda - \mu > 0$. *Let level of risk* $\varepsilon \in (0,1)$ *and time horizon* T *be given.*

(i) If $\mathbf{P}[\tau_{b_0}^{\pi^*_{b_0}} \leq T] \leq \varepsilon$, *then we find* $f(x)$ *such that the value function* $V(x)$ *of the company is* $f(x)$, *and* $V(x) = V(x, b_0) = J(x, \pi^*_{b_0}) = V(x, 0) = f(x)$. *The optimal policy associated with* $V(x)$ *is* $\pi^*_{b_o} = \{A^*_{b_0}(R_\cdot^{\pi^*_{b_o}}), L_\cdot^{\pi^*_{b_o}}\}$, *where* $(R_t^{\pi^*_{b_0}}, L_t^{\pi^*_{b_0}})$ *is uniquely determined by the following SDE with reflection boundary:*

Main Results

Theorem(continue)

$$\begin{cases} dR_t^{\pi^*_{b_o}} = (\mu - (1 - A^*_{b_0}(R_t^{\pi^*_{b_o}}))\lambda)dt + \sigma A^*_{b_0}(R_t^{\pi^*_{b_o}})dW_t - dL_t^{\pi^*_{b_o}}, \\ R_0^{\pi^*_{b_o}} = x, \\ 0 \leq R_t^{\pi^*_{b_o}} \leq b_0, \\ \int_0^\infty I_{\{t:R_t^{\pi^*_{b_o}} < b_0\}}(t)dL_t^{\pi^*_{b_o}} = 0 \end{cases} \tag{9}$$

and $\tau_x^{\pi^*_{b_0}} = \inf\{t : R_t^{\pi^*_{b_0}} = 0\}$. *The optimal dividend level is* b_0. *The solvency of the company is bigger than* $1 - \varepsilon$.

Main Results

Theorem(continue)

(ii) If $\mathbf{P}[\tau_{b_0}^{\pi_{b_0}^*} \leq T] > \varepsilon$*, then there is a unique* $b^* > b_0$ *satisfying* $\mathbf{P}[\tau_{b^*}^{\pi_{b^*}^*} \leq T] = \varepsilon$ *and find* $g(x)$ *such that* $g(x)$ *is the value function of the company, that is,*

$$g(x) = \sup_{b \in \mathfrak{B}} \{V(x,b)\} = V(x,b^*) = J(x,\pi_{b^*}^*) \tag{10}$$

and

$$b^* \in \mathfrak{B}, \tag{11}$$

where

$$\mathfrak{B} := \left\{ b : \mathbb{P}[\tau_b^{\pi_b} \leq T] \leq \varepsilon,\ J(x,\pi_b) = V(x,b) \text{ and } \pi_b \in \Pi_b \right\}.$$

Main Results

Theorem(continue)

The optimal policy associated with $g(x)$ *is* $\pi^*_{b^*} = \{A^*_{b^*}(R_\cdot^{\pi^*_{b^*}}), L_\cdot^{\pi^*_{b^*}}\}$*, where* $(R_\cdot^{\pi^*_{b^*}}, L_\cdot^{\pi^*_{b^*}}\})$ *is uniquely determined by the following SDE with reflection boundary:*

$$\begin{cases} dR_t^{\pi^*_{b^*}} = (\mu - (1 - A^*_{b^*}(R_t^{\pi^*_{b^*}}))\lambda)dt + \sigma A^*_{b^*}(R_t^{\pi^*_{b^*}})dW_t - dL_t^{\pi^*_{b^*}}, \\ R_0^{\pi^*_{b^*}} = x, \\ 0 \leq R_t^{\pi^*_{b^*}} \leq b^*, \\ \int_0^\infty I_{\{t: R_t^{\pi^*_{b^*}} < b^*\}}(t) dL_t^{\pi^*_{b^*}} = 0 \end{cases} \tag{12}$$

and $\tau_x^{\pi_b^*} = \inf\{t : R_t^{\pi_b^*} = 0\}$*. The optimal dividend level is* b^**. The optimal dividend policy* $\pi^*_{b^*}$ *and the optimal dividend* b^* *ensure that the solvency of the company is* $1 - \varepsilon$*.*

Main Results

Theorem(continue)

(iii)

$$\frac{g(x,b^*)}{g(x,b_0)} \leq 1. \tag{13}$$

(iv) Given risk level ε risk-based capital standard $x = x(\varepsilon)$ to ensure the capital requirement of can cover the total given risk is determined by $\varphi^{b^*}(T, x(\varepsilon)) = 1 - \varepsilon$, where $\varphi^b(T, y)$ satisfies

$$\left\{\begin{array}{l} \varphi_t^b(t,y) = \frac{1}{2}[A_b^*(y)]^2\sigma^2\varphi_{yy}^b(t,y) + (\lambda A_b^*(y) - \delta)\varphi_y^b(t,y), \\ \varphi^b(0,y) = 1,\ \textit{for}\ \ 0 < y \leq b, \\ \varphi^b(t,0) = 0, \varphi_y^b(t,b) = 0,\ \textit{for}\ t > 0. \end{array}\right. \tag{14}$$

Main Results

Theorem(continue)

where $f(x)$ is defined as follows: If $\lambda \geq 2\mu$, then

$$f(x) = \begin{cases} f_1(x, b_0) = C_0(b_0)(e^{\zeta_1 x} - e^{\zeta_2 x}), & x \leq b_0, \\ f_2(x, b_0) = C_0(b_0)(e^{\zeta_1 b_0} - e^{\zeta_2 b_0}) + x - b_0, & x \geq b_0. \end{cases} \quad (15)$$

If $\mu < \lambda < 2\mu$, then

$$f(x) = \begin{cases} f_3(x, b_0) = \int_0^x X^{-1}(y)dy, \ x \leq m, \\ f_4(x, b_0) = \frac{C_1(b_0)}{\zeta_1} \exp\left(\zeta_1(x - m)\right) + \frac{C_2(b_0)}{\zeta_2} \exp\left(\zeta_2(x - m)\right), \\ \ m < x < b_0, \\ f_5(x, b_0) = \frac{C_1(b_0)}{\zeta_1} \exp\left(\zeta_1(b_0 - m)\right) + \frac{C_2(b_0)}{\zeta_2} \exp\{\zeta_2(b_0 - m)\} \\ \qquad\qquad + x - b_0, \ x \geq b_0. \end{cases}$$

Main Results

Theorem(continue)

$g(x)$ is defined as follows: If $\lambda \geq 2\mu$, then

$$g(x) = \begin{cases} f_1(x,b), & x \leq b, \\ f_2(x,b), & x \geq b. \end{cases} \tag{17}$$

If $\mu < \lambda < 2\mu$, then

$$g(x) = \begin{cases} f_3(x,b), & x \leq m(b), \\ f_4(x,b), & m(b) < x < b, \\ f_5(x,b), & x \geq b. \end{cases} \tag{18}$$

Main Results

Theorem(continue)

$A^*(x)$ is defined as follows: If $\lambda \geq 2\mu$, then $A^*(x) = 1$ for $x \geq 0$. If $\mu < \lambda < 2\mu$, then

$$A^*(x) = A(x, b_0) := \begin{cases} -\frac{\lambda}{\sigma^2}(X^{-1}(x))X'(X^{-1}(x)), & x \leq m, \\ 1, & x > m, \end{cases} \tag{19}$$

where X^{-1} denotes the inverse function of $X(z)$, and

$$X(z) = C_3(b_0)z^{-1-c/\alpha} + C_4(b_0) - \frac{\lambda - \mu}{\alpha + c}\ln z, \ \forall z > 0, \quad m(b_0) = X(z_1)$$

Main Results

Theorem(continue)

$$\zeta_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 c}}{\sigma^2}, \quad \zeta_2 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 c}}{\sigma^2},$$

$$b_0 = 2\frac{\ln|\zeta_2/\zeta_1|}{\zeta_2 - \zeta_1}, \quad C_0(b_0) = \frac{1}{\zeta_1 e^{\zeta_1 b_0} - \zeta_2 e^{\zeta_2 b_0}}, \Delta = b_0 - m,$$

$$z_1 = z_1(b_0) = \frac{\zeta_1 - \zeta_2}{(-\zeta_2 - \lambda/\sigma^2)e^{\zeta_1 \Delta} + (\zeta_1 + \lambda/\sigma^2)e^{\zeta_2 \Delta}},$$

$$C_1(b_0) = z_1 \frac{-\zeta_2 - (\lambda/\sigma^2)}{\zeta_1 - \zeta_2}, \quad C_2(b_0) = z_1 \frac{\zeta_1 + (\lambda/\sigma^2)}{\zeta_1 - \zeta_2},$$

$$C_3(b_0) = z_1^{1+c/\alpha} \frac{\lambda(c + \alpha(2\mu/\lambda - 1))}{2(\alpha + c)^2}, \quad \alpha = \frac{\lambda^2}{2\sigma^2},$$

$$C_4(b_0) = -\frac{(\lambda - \mu)c}{(\alpha + c)^2} + \frac{(\lambda - \mu)\alpha}{(\alpha + c)^2} \ln C_3(b_0) + \frac{(\lambda - \mu)\alpha}{(\alpha + c)^2} \ln \frac{(\alpha + c)^2}{(\lambda - \mu)c}$$

Economic and financial explanation

- For a given level of risk and time horizon, if probability of bankruptcy is less than the level of risk, the optimal control problem of (7) and (8) is the traditional (3) and (4), the company has higher solvency, so it will have good reputation. The solvency constraints here do not work. This is a trivial case.

Economic and financial explanation

- If probability of bankruptcy is large than the level of risk ε, the traditional optimal policy will not meet the standard of security and solvency, the company needs to find a sub-optimal policy $\pi^*_{b^*}$ to improve its solvency. The sub-optimal reserve process $R_t^{\pi^*_{b^*}}$ is a diffusion process reflected at b^*, the process $L_t^{\pi^*_{b^*}}$ is the process which ensures the reflection. The sub-optimal action is to pay out everything in excess of b^* as dividend and pay no dividend when the reserve is below b^*, and $A^*(b^*, x)$ is the sub-optimal feedback control function. The solvency probability is $1 - \varepsilon$.

Economic and financial explanation

- We proved that the value function is decreasing w.r.t b and the bankrupt probability is decreasing w.r.t. b, so $\pi^*_{b^*}$ will reduce the company's profit, on the other hand, in view of $\mathbb{P}[\tau_{b^*}^{\pi^*_{b^*}} \leq T] = \varepsilon$, the cost of improving solvency is minimal and is $g(x, b_0) - g(x, b^*)$. Therefore the policy $\pi^*_{b^*}$ is the best equilibrium action between making profit and improving solvency.

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- The risk-based capital $x(\varepsilon, b^*)$ to ensure the capital requirement of can cover the total risk ε can be determined by numerical solution of $1 - \varphi^{b^*}(x, b^*) = \varepsilon$ based on (14). The risk-based capital $x(\varepsilon, b^*)$ decreases with risk ε, i.e., $x(\varepsilon, b^*)$ increases with solvency , so does risk-based dividend level $b^*(\varepsilon)$.

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- The premium rate will increase the company's profit.Higher risk will get higher return

8 steps to get solution

- Step 1: Prove the inequality (5) by Girsanov theorem,comparison theorem on SDE,B-D-G inequality.

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- Step 1: Prove the inequality (5) by Girsanov theorem,comparison theorem on SDE,B-D-G inequality.
- Step 2: Prove

Lemma 1

Assume that $\delta = \lambda - \mu > 0$ *and define* $(R_t^{\pi_b^*,b}, L_t^{\pi_b^*})$ *by the following SDE:*

$$\begin{cases} dR_t^{\pi_b^*,b} = (\mu - (1 - A_b^*(R_t^{\pi_b^*,b}))\lambda)dt + \sigma A_b^*(R_t^{\pi_b^*,b})dW_t - dL_t^{\pi_b^*}, \\ R_0^{\pi_b^*,b} = b, \\ 0 \le R_t^{\pi_b^*,b} \le b, \\ \int_0^\infty I_{\{t:R_t^{\pi_b^*,b}<b\}}(t)dL_t^{\pi_b^*} = 0. \end{cases}$$

Then $\lim_{b\to\infty} \mathbf{P}[\tau_b^{\pi_b^*} \le T] = 0.$

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- Step 8: Numerical analysis of PDE by matlab and

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Thank You !